HW 2, N×N & M×N systems

1) $A^n = \underbrace{A \cdot A \cdot \ldots A}_{n}$

$$\begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/4 & \alpha \\ 0 & 1/4 \end{pmatrix}, \cdot \begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & \frac{\alpha}{4} + \frac{\alpha}{2} \\ 0 & \frac{1}{8} \end{pmatrix},$$

$$\cdot \begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/16 & \frac{\alpha}{8} + \frac{\alpha}{8} + \frac{\alpha}{4} \\ 0 & 1/16 \end{pmatrix}, \ldots = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2) x_i ... # of insects initially in chamber "i"

$$\begin{aligned} 0.4x_1 + 0x_2 + 0x_3 + 0.2x_4 &= 12 \\ 0x_1 + 0.4x_2 + 0.3x_3 + 0.2x_4 &= 25 \\ 0x_1 + 0.3x_2 + 0.4x_3 + 0.2x_4 &= 26 \\ 0.6x_1 + 0.3x_2 + 0.3x_3 + 0.4x_4 &= 37 \end{aligned}$$

$\Rightarrow$ $x_1 = 10$

$x_2 = 20$

$x_3 = 30$

$x_4 = 40$

3) a) $\frac{\partial r}{\partial \alpha_0}$

$$r(\alpha_0) = -2\alpha_0 \int_0^1 f\,dx + \int_0^1 \left(\alpha_0^2 + 2\alpha_0\alpha_1 x + 2\alpha_0\alpha_2 x^2 + 2\alpha_0\alpha_3 x^3\right)dx$$

$$\frac{\partial r}{\partial \alpha_0} = -2\int_0^1 f\,dx + 2\alpha_0 \int_0^1 dx + 2\alpha_1 \int_0^1 x\,dx + 2\alpha_2 \int_0^1 x^2 dx + 2\alpha_3 \int_0^1 x^3 dx$$

$$= \left| \begin{array}{l} \int_0^1 \sin \pi x\,dx = -\frac{1}{\pi}\left[\cos \pi x\right]_0^1 = \frac{2}{\pi} = J_0 \\ \int_0^1 dx = 1 = I_0 \quad \int_0^1 x\,dx = \frac{1}{2} = I_1 \\ \int_0^1 x^2 dx = \frac{1}{3} = I_2 \quad \int_0^1 x^3 dx = \frac{1}{4} = I_3 \end{array} \right| = -\frac{4}{\pi} + 2\alpha_0 + \frac{2}{2}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{2}{4}\alpha_3 \stackrel{!}{=} 0$$

$$\Rightarrow \alpha_0 + \frac{1}{2}\alpha_1 + \frac{1}{3}\alpha_2 + \frac{1}{4}\alpha_3 = \frac{2}{\pi}$$

b) $\dfrac{\partial r}{\partial \alpha_{1,2,3}}$

$$r(\alpha_1) = -2\alpha_1 \int_0^1 x f\,dx + \int_0^1 \alpha_1 x(\alpha_1 x + 2\alpha_0 + 2\alpha_2 x^2 + 2\alpha_3 x^3)\,dx$$

$$r(\alpha_2) = -2\alpha_2 \int_0^1 x^2 f\,dx + \int_0^1 \alpha_2 x^2(\alpha_2 x^2 + 2\alpha_0 + 2\alpha_1 x + 2\alpha_3 x^3)\,dx$$

$$r(\alpha_3) = -2\alpha_3 \int_0^1 x^3 f\,dx + \int_0^1 \alpha_3 x^3(\alpha_3 x^3 + 2\alpha_0 + 2\alpha_1 x + 2\alpha_2 x^2)\,dx$$

$$I_4 = \int_0^1 x^4\,dx = \frac{1}{5} \qquad I_5 = \int_0^1 x^5\,dx = \frac{1}{6} \qquad I_6 = \int_0^1 x^6\,dx = \frac{1}{7}$$

$$\left.\begin{aligned} J_1 &= \int_0^1 x \sin \pi x\,dx = \frac{1}{\pi} \\ J_2 &= \int_0^1 x^2 \sin \pi x\,dx = \frac{1}{\pi} - \frac{4}{\pi^3} \\ J_3 &= \int_0^1 x^3 \sin \pi x\,dx = \frac{1}{\pi} - \frac{6}{\pi^3} \end{aligned}\right\} \text{using per partes integration}$$

$$\frac{\partial r}{\partial \alpha_1} = 2(-J_1 + \alpha_0 I_1 + \alpha_1 I_2 + \alpha_2 I_3 + \alpha_3 I_4) \stackrel{!}{=} 0$$

$$\frac{\partial r}{\partial \alpha_2} = 2(-J_2 + \alpha_0 I_2 + \alpha_1 I_3 + \alpha_2 I_4 + \alpha_3 I_5) \stackrel{!}{=} 0$$

$$\frac{\partial r}{\partial \alpha_3} = 2(-J_3 + \alpha_0 I_3 + \alpha_1 I_4 + \alpha_2 I_5 + \alpha_3 I_6) \stackrel{!}{=} 0$$

$$\Rightarrow \overbrace{\begin{bmatrix} I_0 & I_1 & I_2 & I_3 \\ I_1 & I_2 & I_3 & I_4 \\ I_2 & I_3 & I_4 & I_5 \\ I_3 & I_4 & I_5 & I_6 \end{bmatrix}}^{H_4} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \overbrace{\begin{bmatrix} J_0 \\ J_{\cancel{2}1} \\ J_2 \\ J_3 \end{bmatrix}}^{b}$$

$$H_4 \xrightarrow[\text{elim.}]{\text{Gauss.}} U = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1/12 & 4/45 & 1/12 \\ 0 & 0 & -1/180 & -1/120 \\ 0 & 0 & 0 & 1/2800 \end{bmatrix}$$

4)

$$A = \begin{bmatrix} 1 & 2 & 1 & \cancel{0}2 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ \hline \text{Basic} & \text{Free} & \text{Basic} & \text{Free} \end{array} \Longrightarrow \begin{aligned} x_1 &= -2x_2 - x_4 \\ x_3 &= -x_4 \end{aligned}$$

$$\Longrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} x_4$$

5)

$$X = AX + B \Longrightarrow X - AX = B$$

$$(I - A)X = B$$

$$X = (I - A)^{-1} \cdot B$$

$$X = \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{(I-A)^{-1}} \cdot \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}}_{B} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 3 \end{bmatrix}$$

6)

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 4 & 0 & 1 \\ 3 & 5 & 6 & 1 \end{bmatrix}$$

Columns of A independent if and only if

$$A \cdot \vec{c} = \vec{0} \implies \vec{c} = \vec{0}$$

Reducing A to row echelon form,

$$A \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 4 & 0 & -1 \\ 0 & 5 & 6 & -2 \end{bmatrix} \xrightarrow{R_3 - \frac{5}{4} R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 4 & 0 & -1 \\ 0 & 0 & 6 & -3/4 \end{bmatrix}$$

Dividing by pivots for clarity, which yields "reduced" row echelon form,

$$\underbrace{\begin{array}{cccc} u & v & w & y \\ \left[\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & -1/8 \end{matrix}\right. & & & \left.\right] \end{array}}_{A \to E} \underbrace{\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix}}_{\vec{c}} \overset{!}{=} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}} \implies c = \begin{bmatrix} -1 \\ 1/4 \\ 1/8 \\ 1 \end{bmatrix} y$$

$\implies$ non-zero solution found (in fact, infinitely many)

$\implies$ the set is NOT independent.

From the echelon form it is obvious that the 4^{th} column can be written as a combination of first three columns:

$$\begin{bmatrix} 1 \\ -1 \\ -3/4 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \text{ where } \begin{array}{l} a_1 = 1 \\ a_2 = -1/4 \\ a_3 = -1/8. \end{array}$$

The same applies to A.