- **Gaussian elimination**
- **Matrix multiplication**
- **LU factorization**
- **Inverse matrix and Gauss-Jordan elimination**
- **Ill-conditioned matrices and round-off errors**

SUGGESTED READING:

G. Strang, Linear algebra and its applications, Chapter 1

- Reduce $N \times N$ system by repeatedly subtracting multiples of one equation from another equation
- Geometry: intersection of $n$ subspaces ("hyper-planes"), singular cases
- Mechanics: forward elimination (clear out columns below pivots, $A \to U$) and back-substitution. Stop and think when zero pivot encountered (Row exchange? Look *below the zero pivot* for non-zero entry. Failure signifies singularity, i.e. no solution of $\infty$ of solutions.)
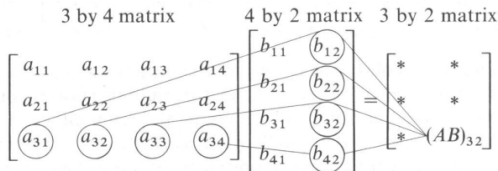- Cost: FE $\approx \frac{1}{3}n^3$, BS $\approx \frac{1}{2}n^2$.

EXAMPLE
$[A|b] \to [U|b]$:

$$
\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \to \begin{bmatrix} \boxed{2} & 1 & 1 & 5 \\ 0 & \boxed{-8} & -2 & -12 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}
$$

# Matrix-Vector and Matrix-Matrix multiplication

- Inner product of two vectors
- $Ax$: by rows, by columns
- $AB$: $(i,j)$ entry is the inner product of $i$th row of $A$ and $j$th column of $B$. Why do it this way? Composition of linear functions, $h = f \circ g$.
- Running time $O(n^2)$, Strassen $O(n^{\log_2 7})$.



3 by 4 matrix      4 by 2 matrix    3 by 2 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \\ * & (AB)_{32} \end{bmatrix}$$

- Associative, distributive, NOT commutative
- Matrix sums and scalar multiplication: done element-wise

EXAMPLE

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix} \textit{ (permutation)}$$

- **Elementary matrix** subtracts a multiple $l$ of row $j$ from row $i$: 1's on the diagonal, $(i,j)$ entry is $-l$.

EXAMPLE

*Gaussian elimination as GFE, applied to A,*

$$GFE = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow GFE \cdot A = U$$

*"Reverse" Gaussian elimination as $E^{-1}F^{-1}G^{-1}$, applied to U,*

$$E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix},$$

$$L \equiv E^{-1}F^{-1}G^{-1} \Rightarrow \boxed{LU = A},$$

*where U contains pivots and L contains multipliers from GE.*

## LU Factorization, comments

▶ A record of elimination steps, gives complete information.

▶ For multiple right hand sides, solve: $Lc = b$, then $Ux = c$, with $\frac{n^2}{2}$ operations each!

▶ How to write $LU$ as $LDU$, where $L$ and $U$ have 1's on the diagonal and $D$ is the diagonal matrix of pivots?

$$
U = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{12}/d_1 & \dots \\ & 1 & u_{23}/d_2 & \dots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}
$$

▶ For a positive definite matrix (pivots positive), $LDU$ reduces to $RR^T$ (Cholesky factorization), where $R$ is upper triangular.

▶ Not all non-singular matrices possess $LU$ factors (leading principal sub-matrices must non-singular).

▶ With the rows reordered in advance, *PA* can be factored into *LU*, where *P* is a permutation matrix (how to find it?), e.g.

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad P_{23} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

EXAMPLE

*Find LU factors of* $A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{bmatrix}$. *To save computer memory, A can be successively overwritten with information in L and U, as GE evolves.*

EXAMPLE

*Use partial pivoting on* $A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 4 & 8 & 12 & -8 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{bmatrix}$ *and determine the LU decomposition PA = LU, where P is the associated permutation matrix. Hint: Adjoin a "permutation counter column" to keep track of row interchanges and remember to permute b as well, if available.*

# Inverses and Transposes

- $A$ invertible, if $A^{-1}A = AA^{-1} = I$
- Simple for 2 by 2 matrix, or diagonal matrix
- $(AB)^{-1} = B^{-1}A^{-1}$, similarly for three or more matrices
- Why bother? $Ax = b \Rightarrow x = A^{-1}b$
- Calculation of $A^{-1}$: <u>Gauss-Jordan elimination</u>
  Instead of stopping at $U$, we continue by subtracting multiples of a row from the rows above, until we reach $I$:

$$\boxed{[A|I] \rightarrow [U|L^{-1}] \rightarrow [I|A^{-1}]}$$

  Not practical though ($n^3$ operations, sensitive to round-off errors).
- Multiple <u>tests for invertibility</u>: independent rows/columns, nonzero pivots/determinant/eigenenvalues, $A^T A$ positive definite, full rank
- Transpose, $(AB)^T$, symmetric matrix

## Ill-conditioned matrices and round-off errors

▶ Finite-precision mathematics, large numbers may swamp small numbers (different scales!).

▶ Ill-conditioned matrix is sensitive to small perturbations. Small pivots VS zero pivots. Condition number $\kappa = \| A \| \| A^{-1} \|$.

EXAMPLE
$A = \begin{bmatrix} 1. & 1. \\ 1. & 1.0001 \end{bmatrix}$. *(ill-conditioned)*
For $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, *while for perturbed* $b = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$!

EXAMPLE
$A' = \begin{bmatrix} 0.0001 & 1. \\ 1. & 1. \end{bmatrix}$. *(well-conditioned)*
For $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, *solve with roundoff to 3 places. Then try* <u>partial pivoting</u>, *i.e. exchange rows so as to maximize the magnitudes of pivots.*