Exam 2, solns

1) (Logistic growth)

$$\frac{dx}{dt} = kx(1000-x), \quad x(0)=1$$

$$\frac{dx}{x(1000-x)} = k\,dt \qquad \text{(SOV)}$$

PF:

$$\frac{1}{1000}\left(\frac{1}{x}+\frac{1}{1000-x}\right) = k\,dt$$

$$\int \frac{1}{x}dx + \int \frac{1}{1000-x}dx = 1000k\int dt$$

$$\ln|x| - \ln|1000-x| = 1000kt + C$$

$$\frac{x}{1000-x} = C_1 e^{1000kt}$$

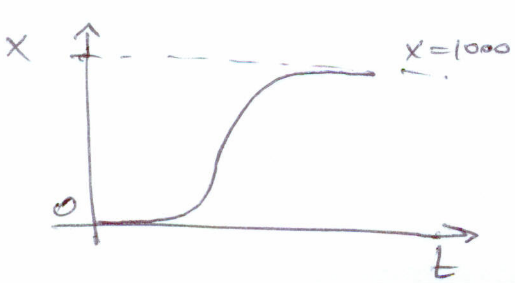
$$x(t) = \frac{1000\,C_1 e^{1000kt}}{1+C_1 e^{1000kt}} = \frac{1000\,C_1}{e^{-1000kt}+C_1}$$

• IC: $x(0)=1$: $1 = \frac{1000C_1}{C_1+1} \Rightarrow C_1 = \frac{1}{999}$

$$\Rightarrow x(t) = \frac{1000}{999e^{-1000kt}+1}$$

• $x(4)=50 \Rightarrow -1000k = \frac{1}{4}\ln\frac{19}{999} \doteq -0.99$

$$x(6) = \frac{1000}{999e^{-5.9}+1} = \underline{276 \text{ students}}$$



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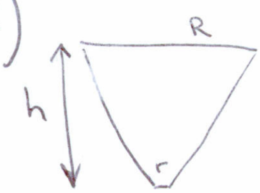
2) 

$R = 6\,ft,\ r = \frac{1}{12}\,ft,\ h = 18\,ft$

$$\frac{R}{h} = \frac{6}{18} = \frac{1}{3} \Rightarrow R = \frac{h}{3}$$

a) $$\frac{dV}{dt} = \frac{A\,dh}{dt}, \quad A = \pi R^2 = \pi \frac{h^2}{9}$$

$$= \underline{\frac{\pi}{9} h^2 \frac{dh}{dt}}$$

b) $$V_{CONE} = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi}{27} h^3$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \frac{\pi}{27} \cdot 3h^2 \frac{dh}{dt} = \underline{\frac{\pi}{9} h^2 \frac{dh}{dt}}$$

$$\frac{dV}{dt} = -4.8\,\pi r^2 \sqrt{h}$$

$$h^{3/2}\,dh = \frac{9}{\cancel{\pi}}(-4.8)\cancel{\pi}\left(\frac{1}{12}\right)^2 dt$$

$$\int h^{3/2}\,dh = -\frac{4.8}{16}\int dt$$

$$\frac{2}{5} h^{5/2} = -\frac{4.8}{16} t + C$$

$$h = \left(-0.75\,t + C\right)^{2/5}, \quad h(0) = 18 \Rightarrow C = 18^{5/2}$$

$$0 = \left(-0.75\,t + 18^{5/2}\right)^{2/5} \Rightarrow \underline{\underline{t = 1833\ s}}$$

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3) $\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \mu (t)$, $x(0) = 0$, $x'(0) = 0$

$\mu \neq \omega$, $F_0 = \text{const.}$

$\lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm i\omega$

$\Rightarrow \underline{x_H = C_1 \cos \omega t + C_2 \sin \omega t}$

x_p : MUC : Assume $\underline{x_p = A \cos \mu t + B \sin \mu t}$

$$x_p'' + \omega^2 x_p = A(\omega^2 - \mu^2)\cos \mu t + B(\omega^2 - \mu^2)\sin \mu t$$

$$= F_0 \sin \mu t$$

$$\Rightarrow \underline{A = 0, \quad B = \frac{F_0}{\omega^2 - \mu^2}}$$

$$\Rightarrow x(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{F_0}{\omega^2 - \mu^2} \sin \mu t$$

IC's : $C_1 = 0$

$C_2 = -\frac{\mu F_0}{\omega(\omega^2 - \mu^2)}$

$$x(t) = \frac{F_0}{\omega(\omega^2 - \mu^2)} \cdot (-\mu \sin \omega t + \omega \sin \mu t), \quad \mu \neq \omega$$

b) $\mu \to \omega$:

$$x(t) = \lim_{\mu \to \omega} F_0 \frac{-\mu \sin \omega t + \omega \sin \mu t}{\omega(\omega^2 - \mu^2)}$$ → L'Hospital's rule $\frac{d}{d\mu}$

$$= F_0 \cdot \lim_{\mu \to \omega} \frac{-\sin \omega t + \omega t \cos \mu t}{-2\omega \mu}$$

$$= \frac{F_0}{2\omega^2} \sin \omega t - \frac{F_0}{2\omega} t \cos \omega t.$$

4) $\cancel{x''+4x}$ $x' + y' + 3x = 15e^{-t}$

$$y'' - 4x' + 3y = 15\sin 2t$$

$x(0) = 35, \quad x'(0) = -48$

$y(0) = 27, \quad y'(0) = -55$

LT:

$$s^2 X - 35s + 48 + sY - 27 + 3X = \frac{15}{s+1}$$

$$s^2 Y - 27s + 55 - 4(sX - 35) + 3Y = \frac{30}{s^2+4}$$

Cramer's rule:

$$X = \frac{\begin{vmatrix} 35s - 21 + \frac{15}{s+1} & s \\ 27s - 195 + \frac{30}{s^2+4} & s^2+3 \end{vmatrix}}{\begin{vmatrix} s^2+3 & s \\ -4s & s^2+3 \end{vmatrix}} = \frac{30s}{s^2+1} - \frac{45}{s^2+9} + \frac{3}{s+1} + \frac{2s}{s^2+4}$$

$$Y = \frac{30s}{s^2+9} - \frac{60}{s^2+1} - \frac{3}{s+1} + \frac{2}{s^2+4}$$

$$x = 30\cos t - 15\sin 3t + 3e^{-t} + 2\cos 2t$$

$$y = 30\cos 3t - 60\sin t - 3e^{-t} + \sin 2t$$