Lecture 9 (2nd order ODEs), Outline

- **Complex numbers, linear independence of functions (Wronskian)**
- **Homogeneous linear DE with constant coefficients; characteristic equations, roots (3 cases)**
- **Nonhomogeneous linear DE with constant coefficients; variation of parameters and undetermined coefficients**
- **Examples**

SUGGESTED READING:

M. Tenenbaum and H. Pollard, Ordinary Differential Equations, Chapter 4

Linear Second-Order ODE

Standard form (nonhomogeneous equation):

$$y'' + p(x)y' + q(x)y = r(x) \tag{**}$$

Reduced form (homogeneous equation):

$$y'' + p(x)y' + q(x)y = 0$$

Existence and Uniqueness Theorem

If $\underline{f_0(x), f_1(x), \cdots, f_n(x) \text{ and } r(x) \text{ are each continuous functions}}$ of x on a common interval I and $\underline{f_n(x) \neq 0}$ when x is in I, then the linear differential equation

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = r(x)$$

has $\underline{\text{one and only one solution}}$ $y = y(x)$.

More on solutions

Form of solutions

If Existence and Uniqueness Theorem satisfied, then the homogeneous linear DE has 2 linearly independent solutions $y_1(x)$ and $y_2(x)$ and their linear combination is also a solution (the general solution, 2-parameter family of solutions), i.e.

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

The general solution (2-parameter family of solutions) of the nonhomogeneous equation is:

$$y(x) = y_h(x) + y_p(x),$$

where $y_p(x)$ is a particular solution of the equation with no arbitrary constants.
$y_1(x)$ and $y_2(x)$ are called a **basis** (or a **fundamental system**) of solutions of the reduced equation.

Linear Independence of Solutions

If p(x) and q(x) are continuous functions of x on an open interval I, then two solutions y_1 and y_2 of (**) are *linearly independent* on I if and only if their *Wronskian W is nonzero* on I:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Example

$$\begin{aligned} W(y_1 = \cos\omega x, y_2 = \sin\omega x) &= \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix} \\ &= \omega\underbrace{\left(\cos^2\omega x + \sin^2\omega x\right)}_{=1} = \omega \end{aligned}$$

$$\begin{aligned} W(y_1 = e^x, y_2 = xe^x) &= \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} \\ &= (x+1)e^{2x} - xe^{2x} = e^{2x} \neq 0 \end{aligned}$$

Second-Order Homogeneous Equations with Constant Coefficients

If the coefficients are functions of x with no restrictions placed on their simplicity, the DE will usually not have solutions expressible in terms of elementary functions. We therefore assume the coefficients are constants.

Standard Form:

$$\boxed{y'' + ay' + by = 0}$$

Postulate a solution of the form $\underline{y = e^{\lambda x}}$

$$\Rightarrow (\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

$$\Rightarrow (\lambda^2 + a\lambda + b) = 0 \qquad \text{(Characteristic equation)}$$

Roots of Characteristic Equation

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right)$$
$$\lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

<u>Three cases</u> depending on sign of $a^2 - 4b$:

Case I. (REAL and DISTINCT) Two real roots if $a^2 - 4b > 0$.
Case II. (REAL and REPEATED) A real double root if $a^2 - 4b = 0$.
Case III. (IMAGINARY) Complex conjugate roots if $a^2 - 4b < 0$.

Case I. Two Distinct Real Roots λ_1 and λ_2

$$y_1 = e^{\lambda_1 x} \qquad y_2 = e^{\lambda_2 x}$$

General Solution (superposition principle):

$$y = \underline{c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

Example

$$y'' - 5y' + 6y = 0$$
$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$
$$\Rightarrow \lambda_1 = 3 \qquad \lambda_2 = 2$$
$$\Rightarrow y(x) = c_1 e^{3x} + c_2 e^{2x}$$

Case II. Real Double Root $\lambda = -a/2$

Problem: The two roots no longer linearly independent.
Solution: Having $y_1 = e^{\lambda x}$, let $y_2 = f(x)e^{\lambda x}$. Plugging-in, we obtain $f'' = 0 \Rightarrow f(x) = c_1 + c_2 x$.

$$y_1 = e^{\lambda x} \qquad y_2 = xe^{\lambda x}, \qquad \text{where } \lambda = -\frac{a}{2}.$$

General Solution:

$$y = \underline{c_1 e^{\lambda x} + c_2 x e^{\lambda x}}$$

Example

$$\begin{aligned} y'' - 6y' + 9y &= 0 \\ \Rightarrow \lambda^2 - 6\lambda + 9 &= 0 \\ \Rightarrow \lambda &= 3 \\ \Rightarrow y(x) &= (c_1 + c_2 x)e^{3x} \end{aligned}$$

$$\lambda_1 = \alpha + i\beta, \qquad \lambda_2 = \alpha - i\beta,$$

$$\text{where } \alpha \equiv -\frac{a}{2} \text{ and } \beta \equiv \sqrt{b - a^2/4}.$$

$$y = d_1 e^{\alpha + i\beta x} + d_2 e^{\alpha - i\beta x}.$$

$$e^{\lambda_1 x} = e^{\alpha x + i\beta x} = e^{\alpha x}(\cos\beta x + i \sin\beta x)$$

$$e^{\lambda_2 x} = e^{\alpha x - i\beta x} = e^{\alpha x}(\cos\beta x - i \sin\beta x)$$

$$\Rightarrow y = \underline{e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)}$$

Example

$$y'' + 4y = 0$$

$$\Rightarrow \lambda^2 + 4 = 0$$

$$\Rightarrow \lambda_1 = +2i \qquad \lambda_2 = -2i$$

$$\Rightarrow y(x) = A \sin 2x + B \cos 2x$$

Particular Integral $y_p(x)$

- Method of Variation of Parameters (MVP)
- Method of Undetermined Coefficients (MUC)
- Method of Inverse Operators

Comments:

- All methods above work for linear DEs of any order (including 1st and 2nd)!
- MVP: General (for both *variable* and *constant* coefficients), BUT more tedious.
- MUC: Faster (no integration needed), BUT only for constant coefficients and special right-hand sides.

Method of Variation of Parameters (MVP)

The solution to the non-homogeneous second-order linear ordinary differential equation:

$$y'' + ay' + by = r(x)$$

is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$
$$+ y_2(x) \underbrace{\int \frac{r(x) y_1(x)}{W(x)} dx}_{u_2(x)} - y_1(x) \underbrace{\int \frac{r(x) y_2(x)}{W(x)} dx}_{u_1(x)}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the corresponding homogeneous equation (r(x)=0), and the Wronskian:

$$W \equiv y_1 y_2' - y_2 y_1'$$

Proof

Postulate:

$$
\begin{aligned}
y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) \\
\Rightarrow y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \\
\Rightarrow y_p'' &= u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' + (u_1'y_1 + u_2'y_2)' \\
\Rightarrow u_1 &\underbrace{\left(y_1'' + py_1' + qy_1\right)}_{=0} + u_2\underbrace{\left(y_2'' + py_2' + qy_2\right)}_{=0} \\
+ &\underbrace{(u_1'y_1' + u_2'y_2')}_{=r(x)} + \underbrace{(u_1'y_1 + u_2'y_2)'}_{=0} + \underbrace{(u_1'y_1 + u_2'y_2)}_{=0} = r(x)
\end{aligned}
$$

$$
\begin{aligned}
u_1'y_1 + u_2'y_2 &= 0, \\
u_1'y_1' + u_2'y_2' &= r(x).
\end{aligned}
$$

Example by MVP

$$y'' - 8y' + 16y = 6xe^{4x}$$

$$y_1 = e^{4x} \qquad y_2 = xe^{4x} \quad W = e^{8x}$$

$$\Rightarrow u_1 = -\int \frac{(xe^{4x})(6xe^{4x})}{e^{8x}} dx = -2x^3$$

$$u_2 = +\int \frac{(e^{4x})(6xe^{4x})}{e^{8x}} dx = 3x^2$$

$$\Rightarrow y = (c_1 + c_2 x + x^3)e^{4x}$$

$$\boxed{y'' + ay' + by = r(x)}$$

<u>CONDITION:</u> Applicable if each term of $r(x)$ has a *finite number of linearly independent derivatives (LIDs)* $\Rightarrow a, x^k, \sin ax, \cos ax$ etc.
<u>PROCEDURE:</u> Differentiate r(x) repeatedly and keep track of the different terms which arise. y_P is written as a linear combination of these terms.
e.g.

$$r(x) = k \sin \omega x \quad \text{or} \quad k \cos \omega x \Rightarrow y_P = K \sin \omega x + M \cos \omega x$$
$$r(x) = kx^n \Rightarrow y_p = a_0 + a_1 x + \cdots + a_p x^n \qquad (n \text{ non-negative integer})$$
$$r(x) = ke^{\alpha x} \Rightarrow y_p = Ce^{\alpha x}$$
$$r(x) = ke^{\alpha x} \cos \omega x \quad \text{or} \quad ke^{\alpha x} \sin \omega x \Rightarrow y_p = e^{\alpha x}(K \cos \omega x + M \sin \omega x)$$

This method cannot be used when repeated differentiation of r(x) does not lead to a finite number of terms, e.g. $\ln x$.

Compare terms of $r(x)$ with those of y_H.
2 cases may arise:

- CASE I.
  IF $r(x)$ and y_H contain different terms;
  THEN y_P is a linear combination (LC) of the terms in $r(x)$ and all its LIDs.
- CASE II.
  IF $r(x)$ contains a term, which (ignoring constants) is x^k a term $u(x)$ of y_H;
  THEN y_P is a LC of $x^{k+r}u(x)$ and all its LIDs, where r is the multiplicity of the root from which $u(x)$ was obtained. For other terms see CASE I.

NOTE: When forming the LC, functions that already appear in y_H may be omitted.

Example (Case I)

$$
\begin{aligned}
y'' + 4y' + 4y &= 6\sin 3x \\
y_h &= (c_1 + c_2 x)e^{-2x} \\
y_p &= a\sin 3x + b\cos 3x \\
y_p'' + 4y_p' + 4y_p &= (-5a - 12b)\sin 3x + (12 - 5b)\cos 3x \\
\Rightarrow -5a - 12b = 6 \qquad & 12a - 5b = 0 \\
\Rightarrow a = -30/169 \qquad & b = -72/169 \\
y &= (c_1 + c_2 x)e^{-2x} - \frac{30}{169}\sin 3x - \frac{72}{169}\cos 3x
\end{aligned}
$$

Example (Case II)

$$y'' - 8y' + 16y = 6xe^{4x}$$

$$\Rightarrow \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$$

$$\Rightarrow y_h = (c_1 + c_2 x)e^{4x}$$

$$y_p = Ax^3 e^{4x}$$

$$y_p' = Ae^{4x}(3x^2 + 4x^3)$$

$$y_p'' = Ae^{4x}(6x + 24x^2 + 16x^3)$$

$$Ae^{4x}\underbrace{[(6x + 24x^2 + 16x^3) - (24x^2 + 32x^3) + 16x^3]}_{=6x} = 6xe^{4x}$$

$$\Rightarrow A = 1$$

$$\Rightarrow y = (c_1 + c_2 x + x^3)e^{4x}$$

MUC examples

Determine the functional form of the particular solution:

1. $y'' + 4y' + 4y = 4x^2 + 6e^x, \qquad y_H = (c_1 + c_2 x)e^{-2x}, \qquad y_P = ?$
2. $y'' - 3y' + 2y = 2xe^{3x} + 3\sin x, \qquad y_H = c_1 e^x + c_2 e^{2x}, \qquad y_P = ?$
3. $y'' - 3y' + 2y = 2x^2 + 3e^{2x}, \qquad y_H = c_1 e^x + c_2 e^{2x}, \qquad y_P = ?$
4. $y'' - 3y' + 2y = xe^{2x} + \sin x, \qquad y_H = c_1 e^x + c_2 e^{2x}, \qquad y_P = ?$
5. $y'' + 4y' + 4y = 3xe^{-2x}, \qquad y_H = (c_1 + c_2 x)e^{-2x}, \qquad y_P = ?$

Example 1 — Homogeneous solution

$$
\begin{aligned}
y'' + 4y' + 3y &= 65\cos 2x \\
\Rightarrow \lambda^2 + 4\lambda + 3 &= 0 \\
\Rightarrow \lambda_1 = -1 \qquad \lambda_2 &= -3 \\
\Rightarrow y_h &= C_1 e^{-x} + C_2 e^{-3x}
\end{aligned}
$$

Example 1 — Particular solution by MVP

$$y_1 = e^{-x} \qquad y_2 = e^{-3x}$$

$$\Rightarrow W = y_1 y_2' - y_2 y_1' = (e^{-x})(-3e^{-3x}) - (e^{-3x})(-e^{-x}) = -2e^{-4x}$$

$$y_p = e^{-3x} \int \frac{(e^{-x})(65\cos 2x)}{(-2e^{-4x})} dx - e^{-x} \int \frac{(e^{-3x})(65\cos 2x)}{(-2e^{-4x})} dx$$

$$= -\frac{65}{2} e^{-3x} \underbrace{\int e^{3x} \cos 2x \; dx}_{\frac{e^{3x}}{13}(3\cos 2x + 2\sin 2x)} + \frac{65}{2} e^{-x} \underbrace{\int e^{x} \cos 2x \; dx}_{\frac{e^{x}}{5}(\cos 2x + 2\sin 2x)}$$

$$= -\cos 2x + 8\sin 2x$$

Example 1 — Particular solution by MUC

For the particular solution by the method of undetermined coefficients, the non-homogeneous term r(x) and its derivatives contain terms only of the form $\cos 2x$ and $\sin 2x$

$$\begin{aligned} y_p &= K\cos 2x + M\sin 2x \\ y_p' &= -2K\sin 2x + 2M\cos 2x \\ y_p'' &= -4K\cos 2x - 4M\sin 2x \end{aligned}$$

substituting into complete ODE and collecting coefficients:

$$\begin{aligned} [-4K + 8M + 3K]\cos 2x + [-4M - 8K + 3M] &= 65\cos 2x \\ \Rightarrow 8M - K = 65 \qquad & -M - 8K = 0 \\ \Rightarrow K = -1 \qquad & M = 8 \end{aligned}$$

Example 2

$$y' - 2y' + y = x^2 + x^{3/2}e^x$$

Homogeneous solution:

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

a repeated root $\lambda = 1$

$$\Rightarrow y_1 = e^x \qquad y_2 = xe^x$$

$$W = y_1 y_2' - y_2 y_1' = (e^x)[e^x(x+1) - (xe^x)(e^x) = e^{2x}$$

Example 2: Particular Solution

Theorem

Let $L(y) = y'' + ay' + by$. If y_1 is the solution of $L(y) = r_1(x)$ and y_2 is the solution of $L(y) = r_2(x)$, then $y_1 + y_2$ is the solution of $L(y) = r_1(x) + r_2(x)$.

For the particular solution, it is convenient to use method of undetermined coefficients for x^2 term and method of variation of parameters for $x^{3/2}e^x$ term:

$$y_{p_1} = k_0 + k_1 x + k_2 x^2$$

Substituting this postulated solution and its derivatives into the ODE

$$(2k_2 - 2k_1 + k_0) + (-4k_2 + k_1) + k_2 x^2 = x^2$$
$$\Rightarrow k_0 = 6 \quad k_1 = 4 \quad k_2 = 1$$

Example 2: Particular Solution

$x^{3/2}e^x$ term:

$$
\begin{aligned}
y_{p_2} &= xe^x \int \frac{(e^x)(35x^{3/2}e^x)}{(e^{2x})}\,dx - e^x \int \frac{(xe^x)(35x^{3/2}e^x)}{(e^{2x})}\,dx \\
&= (35xe^x)\left(\frac{x^{5/2}}{5/2}\right) - (35e^x)\left(\frac{x^{7/2}}{7/2}\right) = 4x^{7/2}e^x \\
y &= (C_1 + C_2 x)e^x + x^2 + 4x + 6 + 4x^{7/2}e^x.
\end{aligned}
$$

EXAMPLE

Find the family of curves with the property that the area of the region bounded by a curve of the family, the x axis, the lines $x = a, x = x$ is proportional to the length of the arc included between these two vertical lines.

EXAMPLE

A body of $T = 180^\circ C$ is immersed in a liquid, which is kept at $T = 60^\circ C$. In 1 minute, T of the immersed body decreases to $120^\circ C$. How long will it take for the body's temperature to decrease to $90^\circ C$?

EXAMPLE

A particle moving on a straight line is attracted to the origin by a force F. If the force of attraction is proportional to the distance x of the particle from the origin, describe the motion that the particle will execute.

EXAMPLE

A capacitor with capacitance $2/1010$ farad, an inductor with coefficient of conductance $1/20$ henry and a resistor with resistance 1 ohm are connected in series. If at $t = 0, i = 0$ and the charge on the capacitor is 1 coulomb, find the charge q and the current i in the circuit due to the discharge of the capacitor when $t = 0.01$ seconds.