Lecture 10 (Laplace transform), Outline

- **Motivation, definition, existence theorem**
- **Fundamental properties (linearity, shifting, convolution etc.)**
- **Applications to differential equations (IVPs, transfer function)**
- **Convenient engineering abstractions: step function and impulse function**
- **Applications to systems of differential equations**

SUGGESTED READING:

E. Kreyszig, Advanced Engineering Mathematics (theory)

M. Spiegel, Theory and Problems of Laplace Transforms (examples)

Overview

- A powerful tool for solving **linear (or linearized)** ODEs and the corresponding IVPs
- Works also for variable coefficients
- The basic strategy of operational calculus:
  Laplace transform **converts ODE to an algebraic equation** for the transformed dependent variable. The algebraic equation is solved explicitly for the transformed dependent variable, and the **solution to the ODE is obtained by finding the inverse Laplace transform**.
- Particularly useful for **right-hand sides (inputs) with discontinuities (unit step), pulses (Dirac's delta) or periodic functions**
- Used in **control theory** to develop simple **input-output models** and analyze influence of disturbances (external variables)

Definition

- An integral transform with an exponential kernel, specifically:

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad s = \gamma + i\omega$$

$$\mathscr{L}^{-1}[F(s)] = f(t) = \lim_{\omega\to\infty} \frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} F(s)e^{st}\,ds$$

Existence theorem places a growth restriction of $f(t)$:

In general if a function f(t) is **piecewise continuous** and of **exponential order**, i.e. there exist constants K, c, and T such that:

$$|f(t)| < Ke^{ct} \qquad \text{for all } t > T$$

then the Laplace transform of the function f(t) exists for all Re s $> c$.

e.g $t^n, e^{\alpha t}$ have Laplace transforms, but not e^{t^n}.

Example 1

$$
\begin{aligned}
\mathscr{L}[e^{\alpha t}] &= \int_0^\infty e^{\alpha t} e^{-st}\, dt \\
&= \int_0^\infty e^{-(s-\alpha)t}\, dt \\
&= - \left. \frac{e^{-(s-\alpha)t}}{(s-\alpha)} \right|_0^\infty \\
&= \frac{1}{(s-\alpha)}, \qquad s > \alpha \\
\mathscr{L}[1] &= \frac{1}{s}, \qquad s > 0
\end{aligned}
$$

Example 2

$$
\begin{aligned}
\mathscr{L}[t^n] &= \int_0^\infty \underbrace{t^n}_{u} \underbrace{e^{-st}\,dt}_{dv} \\
&= -\left.\frac{t^n e^{-st}}{s}\right|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt \\
&= \frac{n}{s}\mathscr{L}[t^{n-1}]
\end{aligned}
$$

Example 2

$$
\begin{aligned}
\mathscr{L}[t^0] &= \frac{1}{s} \\
\mathscr{L}[t^1] &= \frac{1}{s}\mathscr{L}[t^0] = \frac{1}{s^2} \\
\mathscr{L}[t^2] &= \frac{2}{s}\mathscr{L}[t^1] = \frac{2}{s^3} \\
\Rightarrow \mathscr{L}[t^n] &= \frac{n!}{s^{n+1}}
\end{aligned}
$$

Example 3

Using Euler's formula for a complex exponential:

$$
\begin{aligned}
e^{\pm ibt} &= \cos bt \pm i \sin bt \\
\mathscr{L}[\sin(bt)] &= \mathscr{L}\left[\frac{e^{ibt} - e^{-ibt}}{2i}\right] \\
&= \frac{1}{2i}\left(\frac{1}{s - ib} - \frac{1}{s + ib}\right) \\
&= \frac{b}{s^2 + b^2} \\
\mathscr{L}[\cos(bt)] &= \mathscr{L}\left[\frac{e^{ibt} + e^{-ibt}}{2}\right] = \frac{s}{s^2 + b^2}
\end{aligned}
$$

Table of common LTs

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

Some important properties of LTs

Linearity:

$$\mathscr{L}[\alpha f(t)+\beta g(t)] = \alpha F(s)+\beta G(s)$$
$$\mathscr{L}^{-1}[\alpha F+\beta G] = \alpha f(t)+\beta g(t)$$

Change of scale:

$$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

(ii) Transform of derivatives

$$\begin{aligned}
\mathscr{L}[f^{(n)}(t)] &= \int_0^\infty f^{(n)}(t)e^{-st}\,dt \\
&= e^{-st}f^{(n-1)}(t)\Big|_0^\infty + s\int_0^\infty f^{(n-1)}(t)e^{-st}\,dt \\
&= s\mathscr{L}[f^{(n-1)}(t)] - f^{(n-1)}(0) \\
&= s\left[s\mathscr{L}[f^{(n-2)}(t)] - f^{(n-2)}(0)\right] - f^{(n-1)}(0) \\
\Rightarrow \mathscr{L}[f^{(n)}(t)] &= s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) \\
&\quad \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)
\end{aligned}$$

Example 4

$$f(t) = \sin(bt) \quad \dot{f}(t) = b\cos(bt) \quad \ddot{f}(t) = -b^2\sin(bt)$$

$$\mathscr{L}[\ddot{f}(t)] = s^2F(s) - sf(0) - \dot{f}(0)$$

$$\Rightarrow -b^2\mathscr{L}[\sin(bt)] = s^2\mathscr{L}[\sin(bt)] - 0 - b$$

$$\mathscr{L}[\sin(bt)] = \frac{b}{b^2 + s^2}$$

Similarly $f(t) = t \quad \dot{f}(t) = 1$

$$\Rightarrow \mathscr{L}[\dot{f}(t)] = s\mathscr{L}[f(t)] - f(0)$$

$$\Rightarrow \frac{1}{s} = s\mathscr{L}[t] - 0 \Rightarrow \mathscr{L}[t] = \frac{1}{s^2}$$

Partial fractions, review

$$F(s) = \frac{1}{(s+a)(s+b)^2} = \frac{A}{s+a} + \frac{B}{s+b} + \frac{C}{(s+b)^2}$$

$$\Rightarrow A(s+b)^2 + B(s+a)(s+b) + C(s+a) = 1$$

$$s = -b \Rightarrow C = \frac{1}{a-b};\ s = -a \Rightarrow A = \frac{1}{(a-b)^2}$$

Evaluate B by equating coefficients of $s^2 \Rightarrow B = -A$

(iii) Transform of integrals

$$\mathscr{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty \left[\int_0^t f(\tau)d\tau\right] e^{-st}\,dt = F(s)/s$$

$$\mathscr{L}^{-1}[F(s)/s] = \int_0^t f(\tau)d\tau$$

Example 5

Part a:

$$\mathscr{L}^{-1}\left[\frac{1}{s^2-2s}\right]=\mathscr{L}^{-1}\left[\frac{1}{s}\frac{1}{s-2}\right]$$

$$\mathscr{L}^{-1}\left[\frac{1}{s}\frac{1}{s-2}\right]=\int_0^t e^{2\tau}\,d\tau=\frac{1}{2}[e^{2t}-1]$$

Part b:

$$\mathscr{L}^{-1}\left[\frac{1}{s^3-2s^2}\right]=\mathscr{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-2}\right]$$

$$\mathscr{L}^{-1}\left[\frac{1}{s}\frac{1}{s(s-2)}\right]=\frac{1}{2}\int_0^t[e^{2\tau}-1]\,d\tau=\frac{1}{4}[e^{2t}-1-2t]$$

(iv) First shifting theorem (“s-shifting”)

$$\mathscr{L}[e^{\alpha t} f(t)] = F(s-\alpha)$$
$$f(t) = e^{-\alpha t}\mathscr{L}^{-1}[F(s-\alpha)]$$

Example 6

$$\text{Since } \mathscr{L}[\cos 2t] = \frac{s}{s^2+4}, \text{ we have}$$
$$\mathscr{L}[e^{-t}\cos 2t] = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

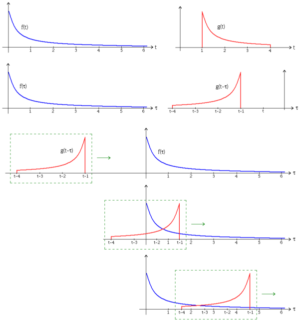
(v) Convolution Theorem:

$$\mathscr{L}[f * g] = F(s)G(s)$$

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau \equiv \int_0^t f(t-\tau)g(\tau)d\tau$$

$$\Rightarrow \mathscr{L}^{-1}[H(s)] = \mathscr{L}^{-1}[F(s)G(s)] = f(t) * g(t)$$

Note: $\mathscr{L}[fg] \neq F(s)G(s)$



Example 7

$$H(s) = \frac{1}{s^3} = \underbrace{\frac{1}{s}}_{F(s)}\underbrace{\frac{1}{s^2}}_{G(s)} \qquad f(t) = 1 \qquad g(t) = t$$

$$\Rightarrow h = f * g = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t (1)(t-\tau)d\tau = \frac{t^2}{2}$$

(vi) Multiplication by t^n and division by t

$$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds} \Rightarrow \text{ derivative of transform}$$

$$\mathscr{L}[t^n f^{(m)}(t)] = (-1)^n \frac{d^n}{ds^n}[s^m F(s) - s^{m-1} f(0) - s^{m-2} \dot{f}(0) - \ldots f^{(m-1)}(0)]$$

$$\mathscr{L}[f(t)/t] = \int_s^\infty F(u)du \Rightarrow \text{ integral of transform}$$

Application: Initial value problems (IVPs), transfer function

$$y'' + ay' + by = r(t) \quad y(0) = K_0 \quad y'(0) = K_1$$

$$[s^2 Y(s) - sy(0) - y'(0)] + a[sY - y(0)] + bY = R$$

$$\Rightarrow Y(s) = R(s)Q(s) + I(s)Q(s)$$

$$Q(s) = \frac{1}{s^2 + as + b} \qquad I(s) = [(s+a)y(0) + y'(0)]$$

$$\Rightarrow \mathbf{y(t)} = \mathbf{r(t)} * \mathbf{q(t)} + \mathbf{i(t)} * \mathbf{q(t)}$$

if $y(0) = 0, y'(0) = 0 \Rightarrow Q = \dfrac{Y}{R} = \dfrac{\mathscr{L}\{output\}}{\mathscr{L}\{input\}}$ **(Transfer function)**

Example 8

$$y'' + \omega^2 y = r(t)$$
$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) + \omega^2 Y(s) = R(s)$$

$$\Rightarrow Y(s) = \frac{1}{s^2+\omega^2} R(s) + \frac{s}{s^2+\omega^2} y(0) + \frac{1}{s^2+\omega^2} y'(0)$$
$$\Rightarrow y(t) = \frac{1}{\omega} \sin(\omega t) * r(t) + y(0)\cos(\omega t) + \frac{y'(0)}{\omega} \sin(\omega t)$$

Exercise: Show that this is the same solution that would be obtained using the method of variation of parameters.

$$\begin{aligned}
y'' + 4y' + 3y &= 65\cos(2t) \\
Y(s) &= Q(s)R(s) + I(s)Q(s) \\
Q(s) &= \frac{1}{s^2+4s+3} = \frac{1}{(s+1)(s+3)} = \frac{1}{2}\left[\frac{1}{s+1} - \frac{1}{s+3}\right] \\
\Rightarrow q(t) &= \frac{1}{2}\left[e^{-t} - e^{-3t}\right] \\
q(t) * r(t) &= \frac{65}{2}\left\{e^{-t}\int_0^t e^{\tau}\cos(2\tau)\,d\tau - e^{-3t}\int_0^t e^{3\tau}\cos(2\tau)\,d\tau\right\} \\
&= \frac{65}{2}\left\{e^{-t}\left[\cos(2\tau) + 2\sin(2\tau)\right]\frac{e^{\tau}}{5}\bigg|_0^t\right. \\
&\quad \left. - e^{-3t}\left[3\cos(2\tau) + 2\sin(2\tau)\right]\frac{e^{3\tau}}{13}\bigg|_0^t\right\} \\
&= 8\sin(2t) - \cos(2t) - \frac{13}{2}e^{-t} - \frac{15}{2}e^{-3t}
\end{aligned}$$

$$
\begin{aligned}
I(s)Q(s) &= \frac{[sy(0)+4y(0)-y'(0)]}{(s+1)(s+3)} \\
&= y(0)\frac{s}{(s+1)(s+3)} + [4y(0)-y'(0)]\frac{1}{(s+1)(s+3)} \\
&= \frac{y(0)}{2}\left[\frac{3}{(s+3)} - \frac{1}{(s+1)}\right] \\
&+ \frac{[4y(0)-y'(0)]}{2}\left[\frac{1}{s+1} - \frac{1}{s+3}\right] \\
&= \frac{[y'(0)-y(0)]}{2}\frac{1}{s+3} + \frac{[3y(0)-y'(0)]}{2}\frac{1}{s+1} \\
\mathscr{L}^{-1}[I(s)Q(s)] &= \frac{[y'(0)-y(0)]}{2}e^{-3t} + \frac{[3y(0)-y'(0)]}{2}e^{-t} \\
\Rightarrow y(t) &= 8\sin(2t) - \cos(2t) + C_1 e^{-t} + C_2 e^{-3t} \\
C_1 &= \frac{[3y(0)-y'(0)]-13}{2} \qquad C_2 = \frac{[y'(0)-y(0)]-15}{2}
\end{aligned}
$$

Example 9 — Slide 3 of 3, comments

- We obtain the same solution with the method of undetermined coefficients, except now we have expressions for C_1 and C_2 explicitly in terms of the initial conditions.
- Once we find the inverse of the transfer function q(t), we can find the response of the system fairly easily for other forcing functions r(t).
- The transfer function $Q(s)$ can be used to obtain information about the behavior of the system. The **roots of the polynomial in s in the denominator of the transfer function** — termed the **poles** — provide information about the **stability and the shape of the system response**.

Step function (Heaviside function)

- Abstraction of a switching mechanism

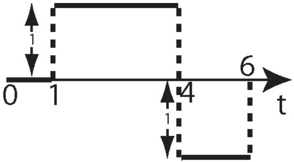
$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

$$\mathscr{L}[u(t-a)] = \int_a^{\infty} e^{-st} dt$$
$$= \frac{e^{-as}}{s}$$

Example 10

$$f(t) = u(t-1) - 2u(t-4) + u(t-6)$$

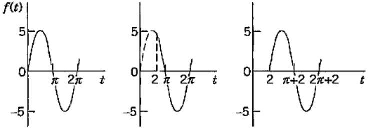
$$f(t) = \begin{cases} 0 & t < 1 \\ +1 & 1 < t < 4 \\ -1 & 4 < t < 6 \\ 0 & t > 6 \end{cases} \qquad \mathscr{L}[f(t)] = \frac{1}{s}[e^{-s} - 2e^{-4s} + e^{-6s}]$$



Second shifting theorem ("t-shifting")

$$\bar{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t > a \end{cases}$$

$$\mathscr{L}[\bar{f}(t)] = \mathscr{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$



(A) $f(t) = 5\sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

Example 11

- Damped mass-spring system under square wave

$$y'' + 3y' + 2y = u(t-1) - u(t-2) \qquad y(0) = 0 = y'(0)$$

$$s^2 Y + 3sY + 2Y = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$Y(s) = F(s)e^{-s} - F(s)e^{-2s}$$

$$F(s) = \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

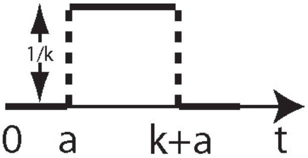
$$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}[F(s)e^{-s} - F(s)e^{-2s}] \\ &= f(t-1)u(t-1) - f(t-2)u(t-2) \end{aligned}$$

Impulse function

- Abstraction of an intense, brief, unit-area pulse

$$f_k(t-a) = \frac{1}{k}[u(t-a) - u(t-k-a)]$$
$$= \begin{cases} \frac{1}{k} & a < t < k+a \\ 0 & \text{otherwise} \end{cases}$$



$$\mathscr{L}[f_k(t-a)] = \frac{1}{k}\int_a^{k+a} e^{-st}dt = -\frac{1}{ks}e^{-st}\Big|_a^{k+a}$$
$$= \frac{1}{ks}[e^{-as} - e^{-(a+k)s}] = e^{-as}\frac{1-e^{-ks}}{ks}$$

Dirac's delta function (as the limit of the impulse fcn)

Definition:

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a) = \begin{cases} \infty & t = a \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^\infty \delta(t-a)dt = 1$$

$$\int_0^\infty \delta(t-a)f(t)dt = f(a) \text{ (sifting property)}$$

$$\mathscr{L}[\delta(t-a)] = \lim_{k\to 0} e^{-as}\frac{1-e^{-ks}}{ks} = e^{-as}$$

Example 12

$$y'' + ay' + by = r(t) = \delta(t-c) \qquad y(0) = 0 \qquad y'(0) = 0$$

$$Y(s) = Q(s)e^{-cs} \qquad Q(s) = \frac{1}{s^2 + as + b}$$

$$= Q(s) \qquad \text{for c=0}$$

Thus the transfer function of a process Q(s) may be interpreted as the response of the process to the Dirac delta function.

Example 13 (on t-shifting)

$$\mathscr{L}^{-1}\left[\frac{e^{-as}}{s(s-2)}\right] = \mathscr{L}^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$$

$$F(s) = \frac{1}{s(s-2)} = \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s}\right]$$

$$\Rightarrow f(t) = \frac{1}{2}\left[e^{-2t} - 1\right]$$

$$\mathscr{L}^{-1}\left[\frac{e^{-as}}{s(s-2)}\right] = \frac{1}{2}\left[e^{-2(t-a)} - 1\right]u(t-a)$$

Example 14

- Alternatively, by convolution:

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{e^{-as}}{s(s-2)}\right] &= \mathscr{L}^{-1}\left[\frac{e^{-as}}{s}\right] * \mathscr{L}^{-1}\left[\frac{1}{s-2}\right] \\
&= u(t-a) * e^{2t} \\
&= \int_0^t u(\tau - a) e^{2(t-\tau)} d\tau \\
&= e^{2t} \int_a^t e^{-2\tau} d\tau \qquad t > a \\
&= \frac{1}{2}\left[e^{-2(t-a)} - 1\right] \qquad t > a \\
&= f(t-a)u(t-a)
\end{aligned}$$

Example 15 — System of ODEs

$$\dot{y}_1 = -0.02y_1 + 0.02y_2 \qquad y_1(0) = 0$$
$$\dot{y}_2 = +0.02y_1 - 0.02y_2 \qquad y_2(0) = 150$$

Taking LT of the ODEs:

$$sY_1 - 0 = -0.02Y_1 + 0.02Y_2$$
$$sY_2 - 150 = +0.02Y_1 - 0.02Y_2$$
$$\Rightarrow Y_2 = -Y_1 + \frac{150}{s} \qquad \text{adding the equations above}$$
$$\Rightarrow sY_1 = -0.02Y_1 - 0.02Y_1 + \frac{3}{s}$$
$$\Rightarrow Y_1 = \frac{3}{s(s+0.04)}$$
$$\Rightarrow y_1(t) = 3\int_0^t e^{-0.04\tau}d\tau = 75 - 75e^{-0.04t}$$
$$\Rightarrow y_2(t) = -y_1(t) + 150 = 75 + 75e^{-0.04t}$$

Example 16 — System of ODEs — 1 of 5

Each of a set of radioactive elements E_1, E_2, E_3 and E_4 disintegrates into the succeeding one at at rate proportional to the number of atoms, except for the end product E_4. If $n_i(t)$ denotes the number of atoms of element E_i present at time t, obtain an equation for $\underline{n_4(t)}$ with M atoms of E_1 present initially.

The coupled system of ODE describing this system is:

$$\begin{aligned}
\dot{n}_1 &= -k_1 n_1 & \qquad n_1(0) &= M \\
\dot{n}_2 &= -k_2 n_2 + k_1 n_1 & \qquad n_2(0) &= 0 \\
\dot{n}_3 &= -k_3 n_3 + k_2 n_2 & \qquad n_3(0) &= 0 \\
\dot{n}_4 &= -k_3 n_3 & \qquad n_4(0) &= 0
\end{aligned}$$

Taking Laplace transforms:

$$\begin{aligned} sN_1 - M &= -k_1 N_1 \\ sN_2 &= -k_2 N_2 + k_1 N_1 \\ sN_3 &= -k_3 N_3 + k_2 N_2 \\ sN_4 &= +k_3 N_3 \end{aligned}$$

$$
\begin{aligned}
N_1 &= \frac{M}{s+k_1} \\
N_2 &= \frac{k_1 N_1}{s+k_2} \\
N_3 &= \frac{k_2 N_2}{s+k_3} \\
N_4 &= \frac{k_3 N_3}{s} \\
&= \frac{k_3}{s} \frac{k_2}{s+k_3} \frac{k_1}{s+k_2} \frac{M}{s+k_1} \\
&= M k_1 k_2 k_3 \left[\frac{C_1}{s+k_1} + \frac{C_2}{s+k_2} + \frac{C_3}{s+k_3} + \frac{C_4}{s} \right]
\end{aligned}
$$

$$C_1 = \frac{1}{k_1(k_2 - k_1)(k_3 - k_1)}$$

$$C_2 = \frac{1}{k_2(k_3 - k_2)(k_1 - k_2)}$$

$$C_3 = \frac{1}{k_3(k_2 - k_3)(k_1 - k_3)}$$

$$C_4 = \frac{1}{k_1 k_2 k_3}$$

$$
\begin{aligned}
\Rightarrow \frac{N_4}{M} = \frac{1}{s} - \frac{k_2 k_3}{(k_2 - k_1)(k_3 - k_1)} \frac{1}{(s + k_1)} \\
- \frac{k_1 k_3}{(k_1 - k_2)(k_3 - k_2)} \frac{1}{(s + k_2)} \\
- \frac{k_1 k_2}{(k_1 - k_3)(k_2 - k_3)} \frac{1}{(s + k_3)}
\end{aligned}
$$

Taking the inverse Laplace transform

$$
\begin{aligned}
\frac{n_4}{M} = 1 - \frac{k_2 k_3 e^{-k_1 t}}{(k_2 - k_1)(k_3 - k_1)} \\
- \frac{k_1 k_3 e^{-k_2 t}}{(k_1 - k_2)(k_3 - k_2)} - \frac{k_1 k_2 e^{-k_3 t}}{(k_1 - k_3)(k_2 - k_3)}
\end{aligned}
$$

- Allow for consumption of A in the tank at a rate -kC_A per unit volume:

$$\underbrace{V\frac{dC_A}{dt}}_{\text{Accumulation}} = \underbrace{FC_{Ai}}_{\text{input}} - \underbrace{FC_A}_{\text{output}} - \underbrace{VkC_A}_{\text{consumption}} \qquad V\text{=liquid volume}$$

$$\Rightarrow \frac{V}{F+Vk}\frac{dC_A}{dt} + C_A = \frac{F}{F+Vk}C_{Ai}$$

$$\Rightarrow \tau\frac{dC_A}{dt} + C_A = KC_{Ai} \qquad \tau = \frac{V}{F+Vk},\ K = \frac{F}{F+Vk}$$

at steady state ($dC_A/dt=0$): $C_{As}=KC_{Ai}$ and we can define a deviation concentration $C_A^* = C_A - C_{As}$:

$$\tau\frac{dC_A^*}{dt} + C_A^* = KC_{Ai}^* \qquad C_A^*(t=0) = 0$$

$$\tau \frac{dC_A^*}{dt} + C_A^* = KC_{Ai}^* \qquad C_A^*(t=0) = 0$$

Thus the dynamics of the process (represented by the ODE) can be seen as modifying the input C_{Ai} (represented by the non-homogeneous term) to produce the output C_A (represented by the dependent variable). Changes to the process input due to changes in operating conditions or process disturbances may be modeled using step and impulse functions.