Lecture 8 (1st order ODEs), Outline

- **Classification of ODEs, directional field**
- **Separation of variables, homogenoues functions**
- **Exact differentials and their physical significance**
- **Integrating factors, variation of parameters, 'shrewd substitutions'**

SUGGESTED READING:

M. Tenenbaum and H. Pollard, Ordinary Differential Equations, Chapters 1 and 2

Definitions

Differential Equation A relationship between changing entities (variables) and rates of change (derivatives).

$$\frac{dy}{dx} = x^2 y - 3 \qquad \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} = 0$$

Solution Any function (explicit or implicit) which does not involve derivatives, and which satisfies identically the differential equation, e.g.

$$y = C_1 \cos(ax) + C_2 \sin(ax) \text{ is a solution of } y'' + a^2 y = 0$$

Definitions

Ordinary Differential Equation (ODE) A differential equation which involves derivatives with respect to only one independent variable.

Partial Differential Equation A differential equation which involves derivatives with respect to two or more independent variables.

Order The order of a differential equation is the order of the highest order derivative which is present.

$$y'' + 2yy' + y = \sin(x) \text{ is a second-order ODE}$$

Degree The power to which the highest derivative term is raised.

$$(y'')^2 = (1 + y')^3 \text{ is of degree two}$$

Definitions

Linear Differential Equation Does not involve product terms among dependent variables and their derivatives. i.e. all coefficients are functions of independent variables or constant. Otherwise the equation is termed **non-linear**.

$$y'' + 3y' + y = e^x \text{ is linear}$$
$$(y'')^2 = (1 + y')^3 \text{ is nonlinear}$$

Homogeneous Differential equation Every single term contains the dependent variables or their derivatives.
Else **non-homogeneous**.

$$y'' + 3y' + y = 0 \text{ is homogeneous}$$
$$y'' + 3y' + y = e^x \text{ is non-homogeneous}$$

General vs particular solution

Under certain conditions (to be detailed in subsequent sections), an n-th order differential equation has a solution with *n* arbitrary constants (**parameters**), and this **general solution**, or more precisely, an **n-parameter family of solutions** is unique. A **particular solution** satisfies the equation and does not contain arbitrary constants. A general solution contains *every* particular solution. The *n* arbitrary constants are determined from initial or boundary conditions.

$$y = A\sin 2x + B\cos 2x$$

is the general solution of the second-order equation

$$y'' = -4y$$

the particular solution satisfying y(0)=0, y'(0)=2 is

$$y = \sin 2x$$

EXAMPLE

Find a differential equation whose 2-parameter family of solutions is

$$y = c_1 e^x + c_2 e^{-x}.$$

[Hint: 2 constants suggest 2nd order equation, thus differentiate twice. The differential equation must not contain any constants, so you need to eliminate them.]

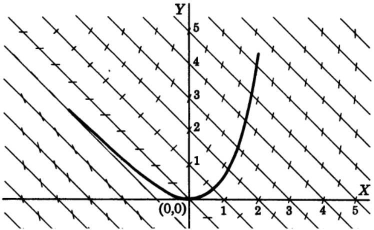
Directional field

Geometric interpretation of the solutions: **integral curves**

<u>EXAMPLE</u>

$y' = x + y \quad \Rightarrow \quad y = e^x - x - 1$

$y \backslash x$	−5	−4	−3	−2	−1	0	1	2	3	4	5
−5	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
−4	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
−3	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2
−2	−7	−6	−5	−4	−3	−2	−1	0	1	2	3
−1	−6	−5	−4	−3	−2	−1	0	1	2	3	4
0	−5	−4	−3	−2	−1	0	1	2	3	4	5
1	−4	−3	−2	−1	0	1	2	3	4	5	6
2	−3	−2	−1	0	1	2	3	4	5	6	7
3	−2	−1	0	1	2	3	4	5	6	7	8
4	−1	0	1	2	3	4	5	6	7	8	9
5	0	1	2	3	4	5	6	7	8	9	10



First-Order Differential Equations

A first-order differential equation of the form:

$$\boxed{\text{y'=f(x,y)}} \quad \Longleftrightarrow \quad N(x,y)\frac{dy}{dx} + M(x,y) = 0$$

can always be expressed in the form:

$$M(x,y)dx + N(x,y)dy = 0$$

Existence and Uniqueness Theorem

The sufficient condition for this equation to have a unique solution which passes through any given point of a region R of the xy plane is that f(x,y) and $\frac{\partial f}{\partial y}$ are **continuous**, real, finite and single-valued in R.

Separable Differential Equations

A first-order differential equation is termed **separable** if it can be expressed in the form:

$$g(y)dy = f(x)dx \quad \Longleftrightarrow \quad \underline{y' = F(x) \cdot G(y)}$$

$$\Rightarrow \int g(y)\,dy = \int f(x)dx + c$$

Reduction of eqn with homogeneous coeffs to separable form

- f(x,y) is **homogeneous of order n** if $f(tx, ty) = t^n f(x, y)$, where $t > 0$ and n is a constant.

Reduction to separable form

The differential equation $M(x, y)dx + N(x, y)dy = 0$ with coefficients $M(x, y)$ and $N(x, y)$ being homogeneous functions of order n can be reduced to a separable form by substitution $y = ux, dy = u \cdot dx + x \cdot du$ (or $x = uy$).

Example

An ODE of the form:

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

can be made separable by the transformation u=y/x:

$$y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx} = g(u)$$
$$\Rightarrow \frac{dx}{x} = \frac{du}{g(u) - u}$$

<u>EXAMPLE</u>

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2} = \frac{(y/x)^2}{(y/x) - 1} = \frac{u^2}{u - 1}$$
$$\Rightarrow \frac{dx}{x} = \frac{du}{u/(u-1)} \Rightarrow u - \ln u = \ln x + c$$

Exact Differential Equations

- A differential expression (form) $M(x,y)dx + N(x,y)dy$ is called an **exact differential** if it is the **total differential du of some function** $u(x,y)$, i.e. if $M(x,y) = \frac{\partial}{\partial x}u(x,y)$ and $N(x,y) = \frac{\partial}{\partial y}u(x,y)$.
- A 1-parameter **family of solutions** of the differential equation $M(x,y)dx + N(x,y)dy = 0$ is then $\underline{u(x,y) = c}$, where c is a constant.
- A **necessary and sufficient condition** for the equation to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

Example

$$xdx + ydy = 0 \qquad \text{is exact}$$
$$-ydx + xdy = 0 \qquad \text{is not exact}$$

Physical Significance of Exact Differentials

- If the total derivative of a function u is exact, its integral has the same magnitude regardless of the path taken.
- In physical chemistry and in thermodynamics, **state functions** (functions whose values depend only on the state and not on the path taken to get to that state), e.g. internal energy, **are exact differentials**.
- **Path functions** (functions whose values depend on the path), e.g. work, **are inexact differentials**.

Exact Differential Equations — Solution

$$\frac{\partial u}{\partial x} = M \Rightarrow u = \int M(x,y)\,dx + k(y)$$

$$N = \frac{\partial u}{\partial y} = \int \frac{\partial M}{\partial y}\,dx + \frac{dk}{dy}$$

$$\Rightarrow \frac{dk}{dy} = N - \int \frac{\partial M}{\partial y}\,dx$$

$$\Rightarrow k(y) = \int \left[N - \int \frac{\partial M}{\partial y}\,dx \right] dy + c$$

Similarly we can show:

$$u = \int N(x,y)\,dy + l(x)$$

$$l(x) = \int \left[M - \int \frac{\partial N}{\partial x}\,dy \right] dx + c$$

Example

$$\frac{dy}{dx} = -\frac{(2xy+3x^2)}{x^2}$$

$$\Rightarrow (2xy+3x^2)dx + x^2dy = 0$$

$$\Rightarrow M = 2xy+3x^2 \qquad N = x^2$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \Rightarrow \text{ Exact}$$

$$\frac{\partial u}{\partial x} = M = 2xy+3x^2 \Rightarrow u = \int (2xy+3x^2)dx + k(y)$$

$$\Rightarrow u = x^2y + x^3 + k(y)$$

$$\frac{\partial u}{\partial y} = N = x^2 = x^2 + \frac{dk}{dy} \Rightarrow \frac{dk}{dy} = 0 \Rightarrow k(y) = c$$

Note that the solution $u(x,y) = C$ defines y as an implicit function of x.

Some recognizable exact DEs

Exact Differential Equation	*Solution*
$y\,dx + x\,dy = 0$	$xy = c$
$2xy\,dx + x^2\,dy = 0$	$x^2y = c$
$y^2\,dx + 2xy\,dy = 0$	$xy^2 = c$
$2xy^2\,dx + 2x^2y\,dy = 0$	$x^2y^2 = c$
$3x^2y^3\,dx + 3x^3y^2\,dy = 0$	$x^3y^3 = c$
$3x^2y\,dx + x^3\,dy = 0$	$x^3y = c$
$y\cos x\,dx + \sin x\,dy = 0$	$y\sin x = c$
$\sin y\,dx + x\cos y\,dy = 0$	$x\sin y = c$
$ye^{xy}\,dx + xe^{xy}\,dy = 0$	$e^{xy} = c$
$\dfrac{dx}{x} + \dfrac{dy}{y} = 0$	$\log(xy) = c$
$\dfrac{y\,dx - x\,dy}{y^2} = 0$	$\dfrac{x}{y} = c$
$-\dfrac{y\,dx - x\,dy}{x^2} = 0$	$\dfrac{y}{x} = c$
$\dfrac{2xy\,dy - y^2\,dx}{x^2} = 0$	$\dfrac{y^2}{x} = c$

Exact Differential Equation	*Solution*
$\dfrac{2xy\,dx - x^2\,dy}{y^2} = 0$	$\dfrac{x^2}{y} = c$
$\dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$	$\operatorname{Arc\,tan}\dfrac{y}{x} = c$
$2\,\dfrac{x\,dy - y\,dx}{x^2 - y^2} = 0$	$\log\dfrac{x+y}{x-y} = c$
$-\dfrac{3x^2y\,dx - x^3\,dy}{x^6} = 0$	$\dfrac{y}{x^3} = c$
$\dfrac{y^2\,dx - 2xy\,dy}{y^4} = 0$	$\dfrac{x}{y^2} = c$
$-\dfrac{y\,dx + x\,dy}{x^2y^2} = 0$	$\dfrac{1}{xy} = c$
$-\dfrac{y^3\,dx - xy^2\,dy}{x^4} = 0$	$\dfrac{y^3}{3x^3} = c$
$\dfrac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = 0$	$\sqrt{x^2 + y^2} = c$
$e^{3x}\,dy + 3e^{3x}y\,dx = 0$	$e^{3x}y = c$

Integrating Factors

- a multiplying factor which will **convert an inexact DE into an exact one**
- no general rule is known to discover it, except for some special types of DEs

If the first-order ODE:

$$Pdx + Qdy = 0 \text{ is not exact, i.e. } \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

We may be able to find an integrating factor F, such that

$$\frac{\partial (FP)}{\partial y} = \frac{\partial (FQ)}{\partial x}$$

$$\underbrace{(FP)}_{M} dx + \underbrace{(FQ)}_{N} dy = du \text{ is exact}$$

Solving special types of ODEs using integrating factor

$$P(x,y)dx + Q(x,y)dy = 0$$

$$A \equiv \frac{\partial P}{\partial y} \qquad B \equiv \frac{\partial Q}{\partial x}$$

(a) ODE can be put in separable form: g(y)dy=f(x)dx ?

(b) $A = B \Rightarrow$ ODE is exact

(c) $A \neq B$:

(i) if $l = \frac{1}{Q}(A - B)$ is only a function of x

$\Rightarrow$ Integrating Factor $F(x) = e^{\int l(x)dx}$

(ii) if $k = \frac{1}{P}(B - A)$ is only a function of y

$\Rightarrow$ Integrating Factor $F(y) = e^{\int k(y)dy}$

(iii) if P=yf(xy) and Q=xg(xy), then $\frac{1}{(xP - yQ)}$ is an I.F.

Example

$$\frac{dy}{dx} = -\frac{2xy}{y^2 - x^2}$$

$$\Rightarrow 2xydx + (y^2 - x^2)dy = 0$$

$$\Rightarrow A = \frac{\partial(2xy)}{\partial y} = 2x \neq \frac{\partial(y^2 - x^2)}{\partial x} = -2x = B$$

$$I = \frac{2x - (-2x)}{y^2 - x^2} \neq I(x)$$

$$k = \frac{-2x - 2x}{2xy} = -\frac{2}{y} = k(y)$$

$$\Rightarrow F = e^{\int(-2/y)dy} = e^{-2\ln y} = y^{-2}$$

$$\Rightarrow x^2 + y^2 = cy$$

Linear First-Order ODE, Integrating Factor

- both the *dependent variable and its derivative are of the first degree*

$$y' + p(x)y = r(x)$$

Methods of solution

1. (faster) Integrating factor ($F(x) = e^{\int p(x)dx}$)
2. (more general) Separation of variables + Variation of parameters
   ($y_N = y_H + y_P, \quad y_H = C\phi(x) \to y_P = C(x)\phi(x)$)

Using the integrating factor $F(x) = e^{\int p(x)dx}$,

$$\frac{d(yF)}{dx} = Fy' + y\underbrace{pF}_{F'} = F(y' + py) = Fr$$

$$yF = \int F(x)r(x)dx + c$$

$$y(x) = \frac{1}{F}\left[\int F(x)r(x)dx + c\right]$$

EXAMPLE

$$\frac{dy}{dx} + 5y = 50$$

$$\textit{Integrating factor } F = e^{\int 5\,dx} = e^{5x}$$

$$\frac{d}{dx}(ye^{5x}) = 50e^{5x} \Rightarrow ye^{5x} = 10e^{5x} + c \Rightarrow y = 10 + ce^{-5x}$$

EXAMPLE

$$\frac{dy}{dt} + \frac{4}{2t+5}y = 10 \qquad y(t=0) = 0$$

$$\begin{aligned} \textit{Integrating factor } F &= e^{\int 4/(2t+5)\,dt} \\ &= e^{2\ln(2t+5)} = e^{\ln(2t+5)^2} = (2t+5)^2 \end{aligned}$$

$$y(2t+5)^2 = \int 10(2t+5)^2\,dt + c = \frac{5}{3}(2t+5)^3 + c$$

$$y = \frac{5}{3}(2t+5) + c(2t+5)^{-2}$$

- Some non-linear equations can be **transformed to the previous cases (linear, exact etc.) by an appropriate substitution**, thus arriving at an exact solution.

EXAMPLE (BERNOULLI EQUATION)

$$y' + p(x)y = g(x)y^{\alpha}, \qquad \textit{where } \alpha \textit{ is a real number.}$$

$$\textit{Substitution:} \quad u(x) = [y(x)]^{1-\alpha} \qquad \textit{yields}$$

$$u' = (1-\alpha)y^{-\alpha}y' = (1-\alpha)y^{-\alpha}(gy^{\alpha} - py)$$

$$= (1-\alpha)(g - py^{1-\alpha})$$

$$u' + (1-\alpha)pu = (1-\alpha)g,$$

a linear first-order ODE in u.

Example

$$
\begin{aligned}
y' - Ay &= By^2 \\
u &= y^{-1} \\
\Rightarrow u' &= -y^{-2}y' = -y^{-2}(By^2 + Ay) \\
\Rightarrow u' &= -B - Ay^{-1} \\
\Rightarrow u' + Au &= -B \qquad \text{linear 1st-order ODE in u} \\
u &= -e^{-Ax}\left[\int Be^{Ax}\,dx + c\right] = -\frac{B}{A} + ce^{-Ax} \\
y &= \frac{1}{u} = \frac{1}{-(B/A) + ce^{-Ax}}
\end{aligned}
$$