Lecture 3, Outline

- **Vector space, subspaces, linear independence, basis, dimension**
- **Modified Gaussian elimination for rectangular systems, echelon forms**
- **Consistency**
- **Linear transformations (stretch, rotation, reflection, projection)**

SUGGESTED READING:

G. Strang, Linear algebra and its applications, Chapter 2

Vector space

- Vector space: a set of vectors, together with rules for vector addition and scalar multiplication (8 axioms)
- Subspace: closed under addition and scalar multiplication
  - Column space $\mathscr{R}(A)$: set of all combinations of the columns
  - Null space $\mathscr{N}(A)$: set of solutions to $Ax = 0$
  - Row space $\mathscr{R}(A^T)$: column space of A^T
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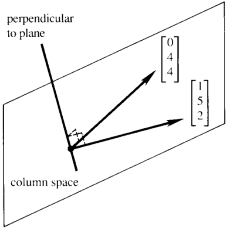
EXAMPLE

$$Ax = b, \quad u \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + v \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$



$Ax = b$ solvable if b can be expressed as a combination of the columns of A, i.e. if b is in $\mathscr{R}(A)$.

Linear independence, basis, dimension

- Independence: only the trivial linear combination ($\sum_{k=1}^{N} c_k v_k$, where $c_1 = c_2 = \cdots = c_k = 0$) gives zero $\Rightarrow$ columns independent iff $\mathscr{N}(A) = 0$, where v's are columns of A (test for independence!).
- Basis: a set of linearly independent vectors that span the space
  - Maximal independent set
  - Minimal spanning set
- Dimension: number of vectors in the basis

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<u>EXAMPLE</u>

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Do the columns of A form a basis of $\mathbf{R}^2$*?*
Is the basis unique? How many bases does a vector space have?

Rectangular systems and echelon forms

- Extend GE from square to rectangular systems: If zero in pivot position encountered, exchange rows or *go on to the next column* $\Rightarrow$ pivots confined to a staircase pattern (echelon form):

$$U = \begin{bmatrix} \circledast & * & * & * & * & * & * & * & * \\ 0 & \circledast & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \circledast & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \circledast \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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EXAMPLE

3×4 *matrix, ignoring at first b,*

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

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Solutions to $Ax = 0$

Let's solve homogeneous system first, $Ax_H = 0$:

- *Basic* and *free* variables correspond to columns with and without pivots, respectively.
- To find *general solution*, express basic variables (r) in terms of free variables ($n - r$).

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What is $\mathcal{N}(A)$? and $\mathcal{N}(U)$? What are their dimensions?
What is $\mathcal{R}(A^T)$ and $\mathcal{R}(U^T)$? What are their dimensions?

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Rank

Number of pivots, number of nonzero rows in echelon form U, number of basic (independent) columns in A.

Solutions to $Ax = b$, consistency

The non-homogeneous system system, $Ax = b$, solved the same way, i.e. use GE to reduce $[A|b]$ to echelon form and solve for basic variables in terms of free variables. But is the system *consistent*, i.e. does it possess at least one solution?

$$Ux_H = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}$$

Consistency

- row $(0 \quad 0 \quad \cdots \quad 0 \quad | \quad \alpha)$, where $\alpha \neq 0$, never appears
- $r[A|b] = r[A]$ (Frobenius)
- b is in the column space of A

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What is the relationship between rank and nullity (dimension of the nullspace)? Hint: For any matrix, the number of basic variables plus the number of free variables must match the total number of columns.

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Dimensions of the 4 fundamental subspaces

Dim($\mathscr{R}(A)$)=r, dim($\mathscr{R}(A^T)$)=r, dim($\mathscr{N}(A)$)=n-r, dim($\mathscr{N}(A^T)$)=m-r.

Existence and uniqueness

- Rectangular matrix can't have an inverse, but it can have right-inverse ($AC = I$) or left-inverse ($BA = I$).
- Inverse exists only when the rank is as large as possible.
- $r \leq \min(m, n)$.

Existence

$Ax = b$ has **at least one solution** if and only if $\mathbf{r} = \mathbf{m}$ (columns span $\mathbf{R}^m$). Then a right inverse C exists ($AC = I_m$). This is possible only if $m \leq n$.

Uniqueness

$Ax = b$ has **at most one solution** if and only if $\mathbf{r} = \mathbf{n}$ (columns independent). Then a left inverse B exists ($BA = I_n$). This is possible only if $m \geq n$.

Only a square matrix can achieve *both* existence and uniqueness.

Linear transformations

For $A \in \mathbf{R}^{\mathbf{m}\times\mathbf{n}}$ and $x \in \mathbf{R}^{\mathbf{n}\times\mathbf{1}}$, Ax is a linear transformation from $\mathbf{R}^{\mathbf{n}}$ into $\mathbf{R}^{\mathbf{m}}$, i.e. $A(\alpha x + y) = \alpha Ax + Ay$.

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EXAMPLE

Stretching by c

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

Rotation around origin by $\theta = 90^\circ$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Reflection in the mirror line $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Projection onto horizontal axis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$