

- ▶ **What are they and how they arise**
- ▶ **Definition by way of example**
- ▶ **Relevant theorems**
- ▶ **Diagonalization**

SUGGESTED READING:

G. Strang, Linear algebra and its applications, Chapter 5

- ▶ Eigenvalues only make sense for square matrices (as does the determinant)
- ▶ Gaussian elimination preserves row space and null space, but not eigenvalues \Rightarrow need for a new technique
- ▶ Eigenvalues and eigenvectors arise naturally in the solution of ordinary differential equations, but they also have a physical origin (natural frequencies and normal modes of a system, respectively). To understand the concepts, however, no knowledge of ODE theory is required.

Definition by example

Consider a coupled system of ODEs (initial value problem, first-order, linear, constant coefficients),

$$\begin{aligned}\frac{dv}{dt} &= 4v - 5w, & v &= 8 \text{ at } t = 0, \\ \frac{dw}{dt} &= 2v - 3w, & w &= 5 \text{ at } t = 0,\end{aligned}$$

Written in a matrix form, we obtain

$$\frac{du}{dt} = Au, \quad u = u_0 \text{ at } t = 0.$$

Recalling that the solution to a scalar equation $\frac{du}{dt} = au$, $u = u_0$ at $t = 0$, is $u(t) = e^{at}u_0$, we assume same functional form for the solution of our system, i.e. $u(t) = e^{\lambda t}x$ (in vector notation), where $x = \begin{bmatrix} y \\ z \end{bmatrix}$. Substituting this into $\frac{du}{dt} = Au$ gives $\lambda e^{\lambda t}x = Ae^{\lambda t}x$, and the cancellation produces

$$Ax = \lambda x.$$

Re-writing as

$$(A - \lambda I)x = 0,$$

and noting that we are interested only in those particular values of λ (eigenvalues) for which there is a *nonzero* solution x (eigenvector), we see that the nullspace of $(A - \lambda I)$ (called *eigenspace*) must be **non-trivial**. That is, $(A - \lambda I)$ must be **singular**.

Characteristic equation

$$\det(A - \lambda I) = 0$$

The polynomial on the left-hand side is the *characteristic polynomial*; its roots are the eigenvalues, its degree is n (for an $n \times n$ matrix A).

Example, continued

Shifting A by λI gives

$$A - \lambda I = \begin{bmatrix} 4 - \lambda & -5 \\ 2 & -3 - \lambda \end{bmatrix},$$

with the characteristic polynomial being

$$(4 - \lambda)(-3 - \lambda) + 10 = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2).$$

To find the eigenvectors, we solve

$$\lambda_1 = -1: \quad (A - \lambda_1 I)x_1 = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\lambda_2 = 2: \quad (A - \lambda_2 I)x_2 = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields $x_1 = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2 = C_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

Using *superposition*, we have a general solution $u = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$.
How to find the constants c_1 and c_2 ?

Theorems on eigenvalues and eigenvectors

- 1 Sum and product of eigenvalues:

$$\sum_{i=1}^n \lambda_i = \sum_{i=1}^n a_{ii} \quad (\text{trace})$$

$$\prod_{i=1}^n \lambda_i = \det(A) \Rightarrow \text{for a non-singular matrix, } 0 \text{ is not an eigenvalue}$$

- 2 *Symmetric* matrix has *real* eigenvalues. Its *eigenvectors* can be chosen *orthonormal*.
- 3 Eigenvectors corresponding to *different* eigenvalues are *linearly independent*.
- 4 For a real matrix with real eigenvalues, the eigenvectors are real.
- 5 For a real matrix with complex eigenvalues, the eigenvalues come in conjugate pairs.

“Nice” vs “defective” matrices, diagonalization

- ▶ Some matrices are “nicer” than others, i.e. easier to work with

EXAMPLE (DIAGONAL AND TRIANGULAR MATRICES)

Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 5 \\ 0 & \frac{3}{4} & 6 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$.

- ▶ How to obtain such nice matrices? **Diagonalization!**

The diagonal form of a matrix

Suppose $n \times n$ matrix A has n linearly independent eigenvectors. Then if they are chosen to be the columns of S , it follows that $S^{-1}AS$ is a diagonal matrix Λ , with eigenvalues of A along its diagonal:

$$S^{-1}AS = \Lambda \quad \Leftrightarrow \quad A = S\Lambda S^{-1}$$

Note: Some matrices are “defective” (not diagonalizable), i.e. they lack a full set of eigenvectors (*algebraic vs geometric multiplicity*).

Example: Markov process

EXAMPLE

Each year 1/10 of the people outside California move in, and 2/10 of the people inside California move out. Does the population eventually approach a steady state? If so, what does the steady state correspond to in terms of eigenvalues and eigenvectors?

[Hint: Form a system of two difference equations for people inside and outside the state. To solve them, diagonalize the matrix A and note that the solution to a difference equation $u_{k+1} = Au_k$ is $u_k = A^k u_0 = (S\Lambda S^{-1})(S\Lambda S^{-1}) \cdots (S\Lambda S^{-1})u_0 = S\Lambda^k S^{-1}u_0$. From the form of u_k try to deduce the limiting state u_∞ .]