Lecture 6 (Eigenvalues/Eigenvectors), Outline

- **What are they and how they arise**
- **Definition by way of example**
- **Relevant theorems**
- **Diagonalization**

SUGGESTED READING:

G. Strang, Linear algebra and its applications, Chapter 5

Introduction

- Eigenvalues only make sense for square matrices (as does the determinant)
- Gaussian elimination preserves row space and null space, but not eigenvalues $\Rightarrow$ need for a new technique
- Eigenvalues and eigenvectors arise naturally in the solution of ordinary differential equations, but they also have a physical origin (natural frequencies and normal modes of a system, respectively). To understand the concepts, however, no knowledge of ODE theory is required.

Definition by example

Consider a coupled system of ODEs (initial value problem, first-order, linear, constant coefficients),

$$\frac{dv}{dt} = 4v - 5w, \qquad v = 8 \text{ at } t = 0,$$

$$\frac{dw}{dt} = 2v - 3w, \qquad w = 5 \text{ at } t = 0,$$

Written in a matrix form, we obtain

$$\frac{du}{dt} = Au, \qquad u = u_0 \text{ at } t = 0.$$

Recalling that the solution to a scalar equation $\frac{du}{dt} = au, u = u_0$ at $t = 0$, is $u(t) = e^{at}u_0$, we assume same functional form for the solution of our system, i.e. $u(t) = e^{\lambda t}x$ (in vector notation), where $x = \begin{bmatrix} y \\ z \end{bmatrix}$. Substituting this into $\frac{du}{dt} = Au$ gives $\lambda e^{\lambda t}x = Ae^{\lambda t}x$, and the cancellation produces

$$\boxed{Ax = \lambda x.}$$

EVAs and EVEs, continued

Re-writing as

$$(A - \lambda I)x = 0,$$

and noting that we are interested only in those particular values of λ (eigenvalues) for which there is a *nonzero* solution x (eigenvector), we see that the nullspace of $(A - \lambda I)$ (called *eigenspace*) must be **non-trivial**. That is, $(A - \lambda I)$ must be **singular**.

Characteristic equation

$$\det(A - \lambda I) = 0$$

The polynomial on the left-hand side is the *characteristic polynomial*; its roots are the eigenvalues, its degree is n (for an $n \times n$ matrix A).

Example, continued

Shifting A by λI gives

$$A - \lambda I = \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix},$$

with the characteristic polynomial being

$$(4-\lambda)(-3-\lambda)+10 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2).$$

To find the eigenvectors, we solve

$$\lambda_1 = -1: \qquad (A-\lambda_1 I)x_1 = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\lambda_2 = 2: \qquad (A-\lambda_2 I)x_2 = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields $x_1 = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 = C_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

Using *superposition*, we have a general solution $\underline{u = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2.}$

How to find the constants c_1 and c_2?

Eigenvalue theorems

Theorems on eigenvalues and eigenvectors

1. Sum and product of eigenvalues:

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} \qquad \text{(trace)}$$

$$\prod_{i=1}^{n} \lambda_i = \det(A) \Rightarrow \text{ for a non-singular matrix, 0 is not an eigenvalue}$$

2. *Symmetric* matrix has *real* eigenvalues. Its *eigenvectors* can be chosen *orthonormal*.
3. Eigenvectors corresponding to *different* eigenvalues are *linearly independent*.
4. For a real matrix with real eigenvalues, the eigenvectors are real.
5. For a real matrix with complex eigenvalues, the eigenvalues come in conjugate pairs.

“Nice” vs “defective” matrices, diagonalization

- Some matrices are “nicer” than others, i.e. easier to work with

  EXAMPLE (DIAGONAL AND TRIANGULAR MATRICES)

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  Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 & 5 \\ 0 & \frac{3}{4} & 6 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$.

- How to obtain such nice matrices? **Diagonalization**!

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  Example (Diagonal and triangular matrices)

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The diagonal form of a matrix

Suppose $n \times n$ matrix A has n linearly independent eigenvectors. Then if they are chosen to be the columns of S, it follows that $S^{-1}AS$ is a diagonal matrix Λ, with eigenvalues of A along its diagonal:

$$\boxed{S^{-1}AS = \Lambda \quad \Leftrightarrow \quad A = S\Lambda S^{-1}}$$

Note: Some matrices are “defective” (not diagonalizable), i.e. they lack a full set of eigenvectors (*algebraic vs geometric multiplicity*).

Example: Markov process

EXAMPLE

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<u>EXAMPLE</u>

Each year $1/10$ *of the people outside California move in, and* $2/10$ *of the people inside California move out. Does the population eventually approach a steady state? If so, what does the steady state correspond to in terms of eigenvalues and eigenvectors?*

[Hint: Form a system of two difference equations for people inside and outside the state. To solve them, diagonalize the matrix A and note that the solution to a difference equation $u_{k+1} = Au_k$ *is* $u_k = A^k u_0 = (S\Lambda S^{-1})(S\Lambda S^{-1})\cdots(S\Lambda S^{-1})u_0 = S\Lambda^k S^{-1}u_0$*. From the form of* u_k *try to deduce the limiting state* u_∞*.]*