Lecture 1, Outline

- **Syllabus (course content, course policy)**
- **Calculus review**

Completing the Square (Quadratic Factorization)

To factor an expression of the form: ax^2+bx+c

1. Factor the coefficient a from the terms involving x
2. Add and subtract the square of half the coefficient of the x term of the monic polynomial

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c \\ &= a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c \\ &= a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right) \end{aligned}$$

$$2x^2+x+1=2\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)+1-\frac{1}{8}=2(x+\frac{1}{4})^2+\frac{7}{8}$$

- Why? Evaluation of integrals, Laplace transforms, ...
- $\int \frac{dx}{2x^2+x+1} = ?$

A function $f(x)$ which is a ratio of two polynomials, with the degree of the polynomial in the numerator $P(x)$ less than the degree of the polynomial in the denominator $Q(x)$, can be expressed in a simpler form that is convenient for many applications using the method of partial fractions.

$$f(x) = \frac{P(x)}{Q(x)}$$

The first step is to factor the denominator as much as possible, then expand in terms of partial fractions (see table below).

$$\frac{3x-1}{x^2-x-6} = \frac{3x-1}{(x+2)(x-3)} = \frac{A}{(x+2)} + \frac{B}{(x-3)}$$
$$\Rightarrow 3x-1 = A(x-3)+B(x+2)$$

as this equation hold for all values of x, use $x=-2$ and $x=3$ to get $A = 7/5$ and $B = 8/5$.

Factor in denominator	Term in partial fraction decomposition
$(ax+b)$	$\frac{A}{(ax+b)}$
$(ax+b)^n$	$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$
(ax^2+bx+c)	$\frac{Ax+B}{(ax^2+bx+c)}$
$(ax^2+bx+c)^n$	$\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$

Show:

$$\frac{2x-3}{x(x^2+1)}=-\frac{3}{x}+\frac{3x+2}{x^2+1}$$

If the degree of $P(x)$ is greater than or equal to that of $Q(x)$, first do polynomial division then decompose the remainder into partial fractions.

Logarithms and Exponentials

The natural logarithm and exponential function are inverse functions:

$$
\begin{aligned}
\ln(e^x) &= e^{\ln x} = x \\
a \cdot \ln x &= \ln x^a \Rightarrow \exp(a \cdot \ln x) = x^a \\
f(x)^{g(x)} &= e^{g(x) \ln f(x)} \\
\ln a + \ln b &= \ln(a \cdot b) \neq \ln a \cdot \ln b \\
\ln a - \ln b &= \ln(a/b) \neq \frac{\ln a}{\ln b} \\
\exp a \cdot \exp b &= \exp(a+b) \neq \exp(a \cdot b) \\
\frac{\exp a}{\exp b} &= \exp(a-b) \neq \exp(a/b)
\end{aligned}
$$

Analogous relations hold for any other base (e.g. base 10).

Trigonometric Identities

- Euler formula, Law of Sines, Law of Cosines
- Double-angle and half-angle formulas

$$
\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(-x) &= -\sin(x) && \text{odd function of x} \\
\cos(-x) &= \cos(x) && \text{even function of x} \\
\sin(x \pm y) &= \sin x \cdot \cos y \pm \cos x \cdot \sin y \\
\cos(x \pm y) &= \cos x \cdot \cos y \mp \sin x \cdot \sin y
\end{aligned}
$$

- To convert an angle in degrees to radians, multiply by $\pi/180$

Differentiation

Linear Operator: $$\frac{d}{dx}(f(x)+g(x)) = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$$

Product Rule: $$\frac{d}{dx}f(x)\cdot g(x) = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

Quotient Rule: $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Power Rule: $$\frac{d}{dx}x^a = ax^{a-1}$$

Composite function: $$\frac{d}{dx}(f(g(x))) = f'(g(x))\cdot g'(x)$$

Derivatives of simple functions

$$\frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}a^x = (\ln a)\cdot a^x$$

$$\frac{d}{dx}xe^{x^2} = e^{x^2} + 2x^2e^{x^2} = e^{x^2}(1+2x^2)$$

- What about $\log_a x, \arcsin x, \arctan x, \ln f(x)$?
- $(\sin x)^{\cos x} = ?$

L'Hopital's Rule

- Use derivatives to evaluate limits of indeterminate forms

If the ratio of two functions is indeterminate:

$$\lim_{x \to a} \frac{f(x)}{g(x)} \to \frac{0}{0} \quad or \quad \frac{\infty}{\infty}$$

L'Hopital's rule can be applied to get:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Multiple applications may be necessary to get a determinate form.

Example

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{0}{0} \Rightarrow \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$$

- $\lim_{x \to 0} \frac{2x+1}{x+4} = ?$

Taylor Series expansion, interval of convergence

Expanding a function around the point $x = x_0$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1} \cdot (x - x_0) + \frac{f''(x_0)}{2} \cdot (x - x_0)^2 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} \cdot (x - x_0)^n$$

The Taylor series for e^x for $x_0 = 0$

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

The Taylor series for $(1 - x)^{-1}$ around $x_0 = 0$

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots$$

The Taylor series for $\sin x$ around $x_0 = 0$

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

Integrals of simple functions

$$\int x^a dx = \begin{cases} \frac{1}{a+1} x^{a+1} + c & a \neq -1 \\ \ln|x| + c & a = -1 \end{cases}$$

$$\int e^x dx = e^x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = +\sin x + c,$$

where c is the constant of integration.

- $\int a^x dx = ?$
- Check to see if differentiating the anti-derivative gives you the integrand.

Basic integration techniques

Note:

$$\int f(x) \cdot g(x)\, dx \neq \int f(x)\, dx \cdot \int g(x)\, dx$$

Substitution:

$$\begin{aligned}\int x e^{x^2}\, dx &= \frac{1}{2} \int e^u\ du \qquad \text{using } u = x^2,\ du = 2xdx \\ &= \frac{1}{2} e^u + c = \frac{1}{2} e^{x^2} + c\end{aligned}$$

Partial fractions expansion

$$\begin{aligned}\int \frac{3x-1}{x^2-x-6} dx &= \frac{7}{5} \int \frac{dx}{x+2} + \frac{8}{5} \int \frac{dx}{x-3} \\ &= \frac{7}{5} \ln(x+2) + \frac{8}{5} \ln(x-3)\end{aligned}$$

Basic integration techniques, cont'd

Per partes (by parts):

- Heuristic, rather than mechanical
- Extends to higher dimensions (1st Green's identity)

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u}\underbrace{\cos x\,dx}_{v'} = \underbrace{x}_{u}\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\underbrace{dx}_{u'}$$

$$= x\sin x + \cos x + c$$

What if we have a definite integral?

For u we usually choose a function that simplifies when differentiated. This does not hold for the following example, but integrating by parts twice gives back the original integral as part of the expression, which can be rearranged to give the solution:

$$\int e^x \cos x = e^x\,\frac{\sin x + \cos x}{2} + c$$

Next time...

- **Gaussian elimination**
- **Matrix multiplication**
- **LU factorization**
- **Inverse matrix and Gauss-Jordan elimination**
- **Ill-conditioned matrices and round-off errors**