HW 6, Solns

①

a) $\mathcal{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\}$

$$\frac{2s-4}{s(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$2s-4 = A\underbrace{(s+1)(s^2+1)}_{s^3+s^2+s+1} + B\underbrace{(s^2+1)s}_{s^3+s} + \overbrace{(Cs+D)\underbrace{s(s+1)}_{s^2+s}}^{Cs^3+Cs^2+Ds^2+Ds}$$

$$\left.\begin{aligned} s^3 &: 0 = A+B+C \\ s^2 &: 0 = A+C+D \\ s^1 &: 2 = A+B+D \\ s^0 &: -4 = A \end{aligned}\right\} \Rightarrow \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{-4}{s} + \frac{3}{s+1} + \underbrace{\frac{s+3}{s^2+1}}_{\frac{s}{s^2+1}+\frac{3}{s^2+1}}\right\} = -4 + 3e^{-t} + \cos t + 3\sin t$$

b) $\mathcal{L}\left\{t\overbrace{\int_0^t \sin\tau\, d\tau}^{f(t)}\right\} = ?$

$$\mathcal{L}\left\{\int_0^t \sin\tau\, d\tau\right\} \underset{\begin{pmatrix} f(t)=\sin t \\ g(t)=1 \end{pmatrix}}{\overset{\text{CONV.}}{=}} \frac{1}{s^2+1} \cdot \frac{1}{s}$$

$$\mathcal{L}\left\{t\cdot\int_0^t \sin\tau\, d\tau\right\} = -\frac{d}{ds}\left(\frac{1}{s^3+s}\right) = \frac{3s^2+1}{s^2(s^2+1)^2}$$

c) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-10s+29}\right\} = e^{5t}\cos 2t + \dfrac{5}{2}e^{5t}\sin 2t$

$$= \frac{s}{(s-5)^2+4} = \frac{s-5}{(s-5)^2+4} + \frac{5}{(s-5)^2+4}$$

d) $\mathcal{L}^{-1}\left\{\underbrace{\dfrac{s+\pi}{s^2+\pi^2}}_{F(s)}e^{-s}\right\} = \begin{cases} f(t-1) & t>1 \\ 0 & t<1 \end{cases}$, $a=1$

$$F(s) = \frac{s}{s^2+\pi^2} + \frac{\pi}{s^2+\pi^2} \Rightarrow f(t) = \cos(\pi t) + \sin(\pi t)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s+\pi}{s^2+\pi^2}e^{-s}\right\} =$$

$$= \cos\pi(t-1)\,\mathcal{U}(t-1) + \sin\pi(t-1)\,\mathcal{U}(t-1)$$

2) $m = \frac{32}{32} = 1$ slug (ENGINsystem, ENGLISH units)

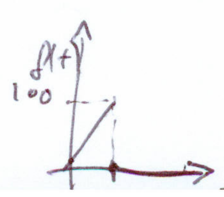
Hooke's Law: $32 = k(2) \Rightarrow k = 16$ lb/ft

$$m\frac{d^2x}{dt^2} + kx = f(t) \quad , \quad \omega^2 = \frac{k}{m} = 16\ ft^{-2}$$

$$f(t) = \begin{cases} 20t & 0 \le t < 5 \\ 0 & t \ge 5 \end{cases}$$

IC's: $x(0) = 0$ (equil. pos.)

$x'(0) = 0$ (rest)



$= 20t - 20t\,\mathcal{U}(t-5)$

- Express $20t\,\mathcal{U}(t-5)$ as $f(t-5)\mathcal{U}(t-5)$ so that we can use the 2^{nd} shifting thm.

$20t = 20(t-5) + 100$

- Taking L.T. of both sides of the eqn,

$$s^2 X + 16X = \mathcal{L}\{20t - \underbrace{20(t-5)\mathcal{U}(t-5)}_{\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)} - 100\mathcal{U}(t-5)\}$$

$$= 20\left[\frac{1}{s^2} - e^{-5s}\cdot\frac{1}{s^2} - 5e^{-5s}\cdot\frac{1}{s}\right]$$

$$\Rightarrow X(s) = \underbrace{\left(\frac{1}{s^2+16}\right)}_{Q(s)}\underbrace{\left(\frac{20}{s^2} - \frac{20e^{-5s}}{s^2} - \frac{100e^{-5s}}{s}\right)}_{R(s)}$$

- $x(t) = r(t) * q(t)$, $r(t) = t - t\,\mathcal{U}(t-5) - 5\,\mathcal{U}(t-5)$

$q(t) = \frac{20}{4}\sin 4t$

OR Partial Fractions:

a) $\mathcal{L}^{-1}\left\{\frac{20}{s^2+16}\cdot\frac{1}{s}\right\} = 5\int_0^t \sin 4\tau\, d\tau = -\frac{5}{4}(\cos 4t - 1) = \frac{5}{4}(1-\cos 4t)$

b) $\mathcal{L}^{-1}\left\{\frac{20}{s^2+16}\cdot\frac{1}{s^2}\right\} = \frac{5}{4}\int_0^t (1-\cos 4\tau)\, d\tau = \frac{5}{4}\left(t - \frac{1}{4}\sin 4t\right)$

$$\Rightarrow x(t) = \frac{5}{4}\left(t - \frac{1}{4}\sin 4t\right) - \frac{5}{4}\left(t-5 - \frac{1}{4}\sin 4(t-5)\right)\mathcal{U}(t-5)$$

$$- \frac{25}{4}\left(1 - \cos 4(t-5)\right)\mathcal{U}(t-5)$$

3) $$y^{(4)} = \underbrace{\frac{W_0}{EI}}_{P_0} \cdot \delta\left(x - \frac{L}{2}\right)$$

$$s^4 Y - s^3 \underbrace{Y(0)}_{0} - s^2 \underbrace{Y'(0)}_{0} - s \underbrace{Y''(0)}_{C_1} - \underbrace{Y'''(0)}_{C_2} = P_0 e^{-\frac{L}{2}s}$$

$$\Rightarrow Y = \frac{C_1}{s^3} + \frac{C_2}{s^4} + \frac{P_0 e^{-L/2\, s}}{s^4}$$

$$\Rightarrow y = C_1 \frac{x^2}{2!} + C_2 \frac{x^3}{3!} + P_0 \frac{\left(x - \frac{L}{2}\right)^3}{3!} U\left(x - \frac{L}{2}\right)$$

$$= \begin{cases} \frac{1}{2} C_1 x^2 + \frac{1}{6} C_2 x^3 & 0 \le x < L/2 \\ \frac{1}{2} C_1 x^2 + \frac{1}{6} C_2 x^3 + \frac{P_0}{6}(x - L/2)^3 & L/2 \le x < L \end{cases}$$

To determine C_1 and C_2:

$$y' = C_1 x + \frac{1}{2} C_2 x^2 + \frac{P_0}{2}(x - L/2)^2 U(x - L/3)$$

$$y'' = C_1 + C_2 x + P_0 (x - L/2) U(x - L/3)$$

$$y''' = C_2 + P_0$$

$$\left.\begin{array}{ll} y''(L) = 0 : & C_1 + C_2 L + P_0 \frac{L}{2} = 0 \\ y'''(L) = 0 : & C_2 + P_0 = 0 \end{array}\right\} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} P_0 L \\ -P_0 \end{pmatrix}$$

$$y = \begin{cases} \frac{W_0}{EI}\left(\frac{L}{4} x^2 - \frac{1}{6} x^3\right) & 0 \le x < L/2 \\ \frac{W_0}{EI}\left(\frac{L}{4} x^2 - \frac{1}{6} x^3 + \frac{1}{6}\left(x - \frac{L}{2}\right)^3\right) & L/2 \le x < L \end{cases}$$

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4) $L_1 s I_2 + RI_2 + RI_3 = \frac{E}{s}$

$L_2 s I_3 + RI_2 + RI_3 = \frac{E}{s}$

$(L_1 s + R) I_2 + RI_3 = \frac{E}{s}$

$RI_2 + (L_2 s + R) I_3 = \frac{E}{s}$

$\Rightarrow I_2 = 10^4 \frac{1}{s(s+900)}$

$I_3 = 8 \times 10^3 \frac{1}{s(s+900)}$

PF: $\frac{1}{s(s+900)} = \frac{1}{900}\left(\frac{1}{s} - \frac{1}{s+900}\right) \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s+900)}\right\} = \frac{1}{900}\left(1 - e^{-900t}\right)$

$\Rightarrow i_2 = \frac{100}{9}\left(1 - e^{-900t}\right)$

$i_3 = \frac{80}{9}\left(1 - e^{-900t}\right)$

$i_1 = i_2 + i_3 = 20\left(1 - e^{-900t}\right)$