Final review

1) Balance chem. reaction

$$C_2H_6 + O_2 \to CO_2 + H_2O$$

$$x_1 C_2H_6 + x_2 O_2 \to x_3 CO_2 + x_4 H_2O$$

$$\left.\begin{aligned} C &: 2x_1 = x_3 \\ H &: 6x_1 = 2x_4 \\ O &: 2x_2 = 2x_3 + x_4 \end{aligned}\right\} \quad \begin{aligned} 2x_1 - x_3 &= 0 \\ 6x_1 - 2x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 6 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} R_2 - 3R_1$$

basic vars

$$\to \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 3 & -2 \end{bmatrix}$$

$x_4 = t \to$ free var.

$$3x_3 - 2t = 0 \Rightarrow x_3 = \frac{2}{3}t$$

$$2x_2 - \frac{4}{3}t - t = 0 \Rightarrow x_2 = \frac{7}{6}t$$

$$2x_1 - \frac{2}{3}t = 0 \Rightarrow x_1 = \frac{1}{3}t$$

Constraint: $\vec{x}$ integers $\Rightarrow t = 6 \Rightarrow$ $x_1 = 2$, $x_2 = 7$, $x_3 = 4$, $x_4 = 6$

2) Tanks in series (Zill, p. 88) FR 2

$$\dot{x}_1 = -\frac{2}{25}x_1 + \frac{1}{50}x_2 \qquad x_1(0) = 25$$

$$\dot{x}_2 = \frac{2}{25}x_1 - \frac{2}{25}x_2 \qquad x_2(0) = 0$$

$$A = \frac{1}{50}\begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix}$$

$$|A - \lambda I| = \frac{1}{50}\begin{vmatrix} -4-\lambda & 1 \\ 4 & -4-\lambda \end{vmatrix} = \frac{1}{50}\left((-4-\lambda)^2 - 4\right)$$

$$= \frac{1}{50}\left(16 + 8\lambda + \lambda^2 - 4\right) = \frac{1}{50}\left(12 + 8\lambda + \lambda^2\right)$$

$$\lambda_{1,2} \begin{cases} -6 \\ -2 \end{cases}$$

a) $\lambda_1 = -6$

$Ax = \lambda x$

$(A - \lambda I)x = 0 \Rightarrow$ eigenspace non-trivial

$$\frac{1}{50}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}_{R_2 - 2R_1} \Rightarrow 2x + y = 0$$

$$x = -\frac{1}{2}y$$

$$\Rightarrow h_1 = C_1\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

b) $\lambda_2 = -2$

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}_{R_2 + 2R_1} \Rightarrow h_2 = C_2\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$x = C_1\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \cdot e^{-6t} + C_2\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{-2t}$$

FR 4

3) $$s^2 X(s) - s\underbrace{\cancel{X(0)}}_{0} - \underbrace{\cancel{x'(0)}}_{v_0} + b\left[s X(s) - \underbrace{\cancel{x(0)}}_{0}\right] = g/s$$

$$(s^2 + bs) X = g/s + v_0$$

$$X = \underbrace{\frac{g}{s(s^2+bs)}}_{F(s)} + \underbrace{\frac{v_0}{s^2+bs}}_{P(s)}$$

$$F(s) = \underbrace{\frac{g}{s}}_{G(s)} \cdot \underbrace{\frac{1}{s^2+bs}}_{H(s)}$$

$$P(s) = \frac{v_0/b}{s} + \frac{-v_0/b}{s+b}$$

$$\downarrow$$

$$p(t) = \frac{v_0}{b}\left(1 - e^{-bt}\right)$$

$$G(s) \to g(t) = g \qquad H(s) \to H(s) = \frac{A}{s} + \frac{B}{s+b}$$

$$A = 1/b, \quad B = -1/b \Rightarrow$$

$$\Rightarrow h(t) = \frac{1}{b} - \frac{1}{b} e^{-bt}$$

$$f(t) = g * h = \int_0^t g\left[\frac{1}{b} - \frac{1}{b} e^{-b(t-\tau)}\right] d\tau$$

$$= \int_0^t g/b \, d\tau - \frac{g}{b}\int_0^t e^{b\tau - bt} \, d\tau$$

$$= \frac{g}{b}\left[\tau\right]_0^t - \frac{g}{b^2}\left[e^{-b\tau - bt}\right]_0^t$$

$$= \frac{g}{b} t - \frac{g}{b^2}\left(1 - e^{-bt}\right) *$$

$$x(t) = f(t) + p(t) = \underline{\underline{\frac{1}{b^2}\left[bgt + (1 - e^{-bt})(bv_0 - g)\right]}}$$

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4) $$\frac{d^2c}{dx^2} - \frac{dc}{dx} - 6c = 0$$

BC's : $x = 0$: $\frac{dc}{dx} = c - 1$

$x = 1$: $\frac{dc}{dx} = 0$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0 \Rightarrow \lambda_{1,2} \begin{cases} 3 \\ -2 \end{cases} \Rightarrow c = c_1 e^{3x} + c_2 e^{-2x}$$

$$\Rightarrow \frac{dc}{dx} = 3c_1 e^{3x} - 2c_2 e^{-2x}$$

$x = 0$: $3c_1 - 2c_2 = c_1 + c_2 - 1$

$$2c_1 - 3c_2 = -1$$

$x = 1$: $3c_1 e^3 - 2c_2 e^{-2} = 0$

$$Ac = b : \begin{bmatrix} 2 & -3 \\ 3e^3 & -2e^{-2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Using Cramer's rule,

$$c_1 = \frac{\begin{vmatrix} -1 & -3 \\ 0 & -2e^{-2} \end{vmatrix}}{-4e^{-2} + 9e^3} = \frac{2e^{-2}}{-4e^{-2} + 9e^3} = \frac{2}{9e^5 - 4}$$

$$c_2 = \frac{\begin{vmatrix} 2 & -1 \\ 3e^3 & 0 \end{vmatrix}}{-4e^{-2} + 9e^3} = \frac{3e^3}{-4e^{-2} + 9e^3} = \frac{3e^5}{9e^5 - 4}$$

$$c(x) = \frac{1}{9e^5 - 4}\left[2e^{3x} + 3e^{5-2x}\right]$$