# Exam 1, solns

1) $[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 2 & 4 & 0 & 1 & b_3 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 2 & 4 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 0 & b_2 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & -5 & b_3 - 2b_1 \\ 0 & 0 & 0 & 0 & b_2 \end{array}\right]$$

$$\begin{array}{cccc} w & x & y & z \\ B & F & F & B \end{array}$$

a) $w + 2x + 3z = 0 \Rightarrow w = -2x$

$-5z = 0 \Rightarrow z = 0$

$$u_H = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2x \\ x \\ y \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{u_1} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{u_2} y$$

$\Rightarrow N(A) = \text{span}\{u_1, u_2\}$

b) Solution exists only if $\underline{b_2 = 0}$. Then,

$-5z = b_3 - 2b_1 \Rightarrow z = \frac{2}{5}b_1 - \frac{1}{5}b_3$

$w + 2x + 3z = b_1 \Rightarrow w = -\frac{1}{5}b_1 + \frac{3}{5}b_3 - 2x$

$$u = \underbrace{\begin{bmatrix} -\frac{1}{5}b_1 + \frac{3}{5}b_3 \\ 0 \\ 0 \\ \frac{2}{5}b_1 - \frac{1}{5}b_3 \end{bmatrix}}_{u_p} + u_H$$

c) $r(A) = 2$

2) 
$$\begin{aligned} P &: \quad 2x + y + z = 10 \\ K &: \quad 3x + 3y = 9 \\ N &: \quad 5x + 4y + z = 19 \end{aligned}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 3 & 0 \\ 5 & 4 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 9 \\ 19 \end{bmatrix}$$

$$[A|b] \xrightarrow{G.E.} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 3/2 & -3/2 & -6 \\ 0 & 3/2 & -3/2 & -6 \end{array}\right] \begin{array}{l} \\ R_2 - \frac{3}{2}R_1 \\ R_3 - \frac{5}{2}R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 3/2 & -3/2 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$
$$\begin{array}{ccc} x & y & z \end{array}$$

$\Rightarrow$ zero pivot $\Rightarrow$ matrix singular $\Rightarrow$ no solution or infinitely many solutions

$$3y - 3z = -16 \quad \Rightarrow y = z - 4$$
$$2x + y + z = 10 \quad \Rightarrow x = 7 - z$$

$$u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7-z \\ z-4 \\ z \end{bmatrix} \begin{array}{l} \Rightarrow z \leq 7 \\ \Rightarrow z \geq 4 \end{array} \Bigg\} \; z \in \{4, 5, 6, 7\}$$

$\Rightarrow$ Taking into account the constraints,

$$u = \left\{ \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix} \right\}.$$

3) $a_1 = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$ $a_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ $\to q_1, q_2$

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{a_1}{7}$$

$$a_2' = a_2 - (q_1^T a_2) q_1 \quad , q_1^T a_2 = \tfrac{1}{7}[4\ 5\ 2\ 2]\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = 2$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - 2 \cdot \frac{1}{7}\begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix}$$

$$\|a_2'\| = \sqrt{\tfrac{1}{7^2}(1 + 3\times 16)} = 1$$

$$q_2 = \frac{a_2'}{\|a_2'\|} = a_2'$$

Fourier expansion : $a_1 = (q_1^T a_1)$

$$a_2 = (q_1^T a_2) q_1 + (q_2^T a_2) q_2$$

Written in terms of QR factors,

$$\underbrace{\begin{bmatrix} a_1 & a_2 \end{bmatrix}}_{A\ (4\times 2)} = \underbrace{\begin{bmatrix} q_1 & q_2 \end{bmatrix}}_{Q\ (4\times 2)} \cdot \underbrace{\begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ & q_2^T a_2 \end{bmatrix}}_{R\ (2\times 2)}$$

$$\overbrace{\begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 2 & 0 \\ 2 & 0 \end{bmatrix}} = \overbrace{\begin{bmatrix} 4/7 & -1/7 \\ 5/7 & 4/7 \\ 2/7 & -4/7 \\ 2/7 & -4/7 \end{bmatrix}} \cdot \overbrace{\begin{bmatrix} 7 & 2 \\ 0 & 1 \end{bmatrix}}$$

4)

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 - 2\lambda = -\lambda(\lambda^2+2) =$$

$$= -\lambda(\lambda+\sqrt{2}\,i)(\lambda-\sqrt{2}\,i)$$

a) $\lambda_1 = 0$ :

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x - z = 0 \\ y = 0 \end{matrix} \Rightarrow h_1 = c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

b) $\lambda_2 = \sqrt{2}\,i$ :

$$\begin{bmatrix} -\sqrt{2}i & -1 & 0 \\ 1 & -\sqrt{2}i & -1 \\ 0 & 1 & -\sqrt{2}i \end{bmatrix} = \sqrt{2}\begin{bmatrix} -i & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -i & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -i \end{bmatrix} \xrightarrow[R_2 - 1/\sqrt{2}\, i R_1]{G.E.}$$

$$\longrightarrow \sqrt{2}\begin{bmatrix} -i & -1/\sqrt{2} & 0 \\ 0 & -1/2\, i & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -i \end{bmatrix} \xrightarrow{R_3 - \frac{2}{\sqrt{2}} i R_1} \begin{bmatrix} -i & -1/\sqrt{2} & 0 \\ 0 & -1/2\, i & -1/\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}\sqrt{2}$$

$$\Rightarrow \quad y = \sqrt{2}\, i\, x$$

$$-\frac{1}{2} i y - \frac{1}{\sqrt{2}} z = 0 \Rightarrow z = -\frac{\sqrt{2}}{2} i y$$

$$\Rightarrow h_2 = c_2 \cdot \begin{bmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{bmatrix}$$

c) $\lambda_3 = -\sqrt{2}\, i$ : $\quad h_3 = c_3 \cdot \begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix}$

d) $$u = c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \underbrace{e^{\sqrt{2}it}}_{1} \cdot \begin{bmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{bmatrix} + c_3 \underbrace{e^{-\sqrt{2}it}}_{1} \cdot \begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix}$$

$$e^{ix} = \cos x + i \sin x \quad \text{(Euler formula)}$$

$$e^{\sqrt{2}t\, i} = \cos \sqrt{2}t + i \sin \sqrt{2}\, t = 1$$

$$e^{-\sqrt{2}t\, i} = \cos \sqrt{2}t - i \sin \sqrt{2}\, t = 1$$

at $t = T$: $\sqrt{2}t \pm 2k\pi = 0$

$\Rightarrow$ for $t > 0$: $\underline{t = \frac{2}{\sqrt{2}} k\pi}$, $k = 1, 2, \ldots$