HW 1, Calculus Review

1. a) $\lim\limits_{x\to\infty} (e^x \operatorname{arccotg} x)$

$$= \lim_{x\to\infty} \frac{\operatorname{arccotg} x}{e^{-x}} = \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-e^{-x}} = \lim_{x\to\infty} \frac{e^x}{1+x^2} = \lim_{x\to\infty} \frac{e^x}{2x} =$$

$$= \lim_{x\to\infty} \frac{e^x}{2} = \underline{\underline{\infty}}$$

b) $$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right) = \lim_{x\to 0^+} \frac{\ln(1+x) - x}{x \ln(1+x)} = \lim_{x\to 0^+} \frac{\frac{1}{1+x} - 1}{\ln(1+x) + \frac{x}{1+x}} =$$

$$= \lim_{x\to 0^+} \frac{\frac{-1}{(1+x)^2}}{\frac{1}{1+x} + \frac{1}{(1+x)^2}} = \underline{\underline{-\frac{1}{2}}}$$

2. a) $$\int e^{2x} \cos x \, dx \overset{\text{PER PARTES}}{=} \left| \begin{matrix} u' = \cos x & u = \sin x \\ v = e^{2x} & v' = 2e^{2x} \end{matrix} \right| =$$

$$= e^{2x} \sin x - \int 2e^{2x} \sin x \, dx = \left| \begin{matrix} u' = \sin x & u = -\cos x \\ v = 2e^{2x} & v' = 4e^{2x} \end{matrix} \right| =$$

$$= e^{2x} \sin x + 2e^{2x} \cos x - 4 \int e^{2x} \cos x \, dx$$

$$\Rightarrow 5 \int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x$$

$$\Rightarrow \int e^{2x} \cos x \, dx = \underline{\underline{\frac{1}{5}\left[e^{2x} \sin x + 2e^{2x} \cos x\right]}}$$

b) $$\int \sqrt{1-t^2} \, dt \overset{\text{SUBSTIT.}}{=} \left| \begin{matrix} t = \sin x \\ dt = \cos x \, dx \end{matrix} \right| = \int \sqrt{1 - \sin^2 x} \cos x \, dx =$$

$$= \int \sqrt{\cos^2 x} \cos x \, dx = \int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} dx =$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x = \frac{1}{2} x + \frac{1}{4} 2 \sin x \cos x = \underline{\underline{\frac{1}{2} \arcsin t + \frac{1}{2} t \sqrt{1-t^2}}}$$

→

c) $\int \frac{x}{x^3-1}\,dx$

Partial fraction expansion : $\frac{x}{x^3-1} = \frac{x}{(x-1)\underbrace{(x^2+x+1)}} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

doesn't have real roots, so don't factor any further

$x \cancel{A} = A(x^2+x+1) + (Bx+C)(x-1)$

$$\left.\begin{aligned} x^2 &: 0 = A+B \\ x^1 &: 1 = A-B+C \\ x^0 &: 0 = A-C \end{aligned}\right\} \quad \begin{aligned} A &= \tfrac{1}{3} \\ B &= -\tfrac{1}{3} \\ C &= \tfrac{1}{3} \end{aligned}$$

$$\Rightarrow I \equiv \int\left(\frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}\right)dx = ? = \frac{1}{3}\Big[\underbrace{\int\frac{1}{x-1}\,dx}_{I_1} + \underbrace{\int\frac{1-x}{x^2+x+1}\,dx}_{I_2}\Big]$$

- $I_1 = \ln|x-1| + C$

Using hint, $I_2 = U\cdot\overbrace{\int\frac{2x+1}{x^2+x+1}}^{I_{21}} + V\cdot\overbrace{\int\frac{1}{x^2+x+1}}^{I_{22}}$

Equating powers of x in the numerators,

$$\left.\begin{aligned} x^1 &: -1 = U\cdot 2 \\ x^0 &: \ 1 = U+V \end{aligned}\right\} \quad \begin{aligned} U &= -\tfrac{1}{2} \\ V &= \tfrac{3}{2} \end{aligned}$$

- $I_{21} = \ln(x^2+x+1)$ $\quad\left[\int\frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C\right]$

- $I_{22} = \int\frac{dx}{\underbrace{(x+\frac{1}{2})^2 - \left(\sqrt{\frac{3}{4}}\right)^2}} = \frac{2}{\sqrt{3}}\,\text{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$ $\quad\left[\int\frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C\right]$

obtained by completing the square of x^2+x+1

Summing up the individual integrals,

$$I = \frac{1}{3}\ln|x-1| + \left(-\frac{1}{6}\right)\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\,\text{arctg}\,\frac{2x+1}{\sqrt{3}} + C'$$

3. Hemispherical surface parametrized by $z=\sqrt{a^2-x^2-y^2}=f(x,y)$

$$I=\iint_S \delta(y^2+z^2)\,dS=\iint_R \left|\frac{\partial r}{\partial x}\times\frac{\partial r}{\partial y}\right|\delta(y^2+z^2)\,dx\,dy,$$

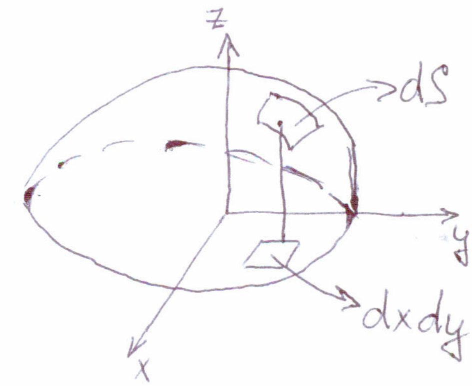
where R is the circular region $x^2+y^2\leq a^2$, and

$\underline{r}=(x,y,z)=(x,y,f(x,y))$.

$$\left|\frac{\partial r}{\partial x}\times\frac{\partial r}{\partial y}\right|=\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}=\sqrt{\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}+1}$$

$$=\frac{a}{\sqrt{a^2-x^2-y^2}}=\frac{a}{z}$$

Thus $I=\delta a\iint_R \frac{a^2-x^2}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$



Using polar coordinates,

$$I=\delta a\int_0^{2\pi}d\theta\underbrace{\int_0^a\frac{a^2-r^2\cos^2\theta}{\sqrt{a^2-r^2}}\,r\,dr}_{I_1}$$

$$I_1=a^2\underbrace{\int\frac{r}{\sqrt{a^2-r^2}}\,dr}_{I_{11}} - \cos^2\theta\underbrace{\int\frac{r^3}{\sqrt{a^2-r^2}}\,dr}_{I_{12}}$$

- $I_{11}\overset{\text{TABLES}}{=}-\sqrt{a^2-r^2}$, $[I_{11}]_0^a=a$

- $I_{12}\overset{\text{SUBSTIT.}}{=}\left|\begin{matrix}r^2=u\\2r\,dr=du\end{matrix}\right|=\frac{1}{2}\int\frac{u}{\sqrt{a^2-u}}\,du\overset{\text{TABLES}}{=}\frac{1}{\cancel{2}}\left[-\frac{\cancel{2}}{3}\left(u\sqrt{a^2-u}-a^2\underbrace{\int\frac{du}{\sqrt{a^2-u}}}_{-2\sqrt{a^2-u}}\right)\right]$

$$=-\frac{1}{3}\left(u\sqrt{a^2-u}+2a^2\sqrt{a^2-u}\right)=-\frac{1}{3}\sqrt{a^2-r^2}\,(2a^2+r^2),$$

$[I_{12}]_0^1=\frac{2}{3}a^3$

$\Rightarrow I_1=a^3-\cos^2\theta\cdot\frac{2}{3}a^3$

$$\Rightarrow I = \sigma a \int_0^{2\pi} \left(a^3 - \frac{2a^3}{3} \cos^2\theta\right) d\theta = \frac{4\pi\sigma a^4}{3}.$$

Since $M = 2\pi a^2 \cdot \sigma$, we have $\underline{\underline{I = \frac{2}{3} M a^2}}$.