Lecture 5 (Determinants), Outline

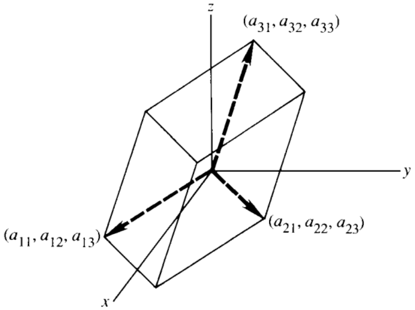
- **Usefulness**
- **Properties and Definition**
- **Applications: Inverse Matrix and Cramer's Rule**
- **Examples**

SUGGESTED READING:

G. Strang, Linear algebra and its applications, Chapter 4

Usefulness

- Test for invertibility: **If determinant of A is zero, then A is singular.**
- **Volume of a parallelpiped** (edges come from rows/columns of A)



- **Formula for pivots**: determinant = $\pm$ (product of pivots)

Properties

1. Determinant changes sign when two rows are exchanged.
2. $\det(I) = 1$.
3. If two rows of A are equal, $\det(A) = 0$.
4. If A contains a zero row, $\det(A) = 0$.
5. Subtracting a multiple of one row from another row leaves the determinant unchanged.
6. If A is triangular, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.
7. $\det(AB) = \det(A)\det(B)$.
8. $\det(A^T) = \det(A)$.

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

$n!$ permutations of rows/columns

Definition

- Two common definitions: expansion in cofactors (Laplace) or permutations of elements (Leibniz)

Expansion in Cofactors

The **determinant** of A is a combination of row i and the cofactors of row i (or, equivalently, column j and the cofactors of column j):

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

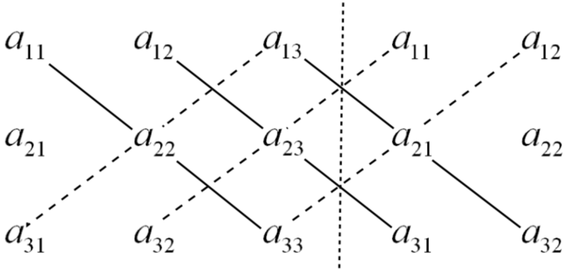
The **cofactor** A_{ij} is the determinant of M_{ij} with the correct sign:

$$A_{ij} = (-1)^{i+j}\det(M_{ij}).$$

The **minor** M_{ij} is formed by deleting row i and column j of A.

Sarrus' Rule

- Based on Lebniz definition of determinant
- A useful memorization scheme to calculate determinants of 3×3 matrices



Applications: Inverse matrix

Writing the cofactor expansion in matrix form, we obtain

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}}_{A_{\text{cof}}} = \underbrace{\begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix}}_{\det(A) I}$$

Multiplying the above by $\frac{A^{-1}}{\det(A)}$, we obtain a formula for A^{-1}:

Inverse matrix

The entries of A^{-1} are the cofactors of A, *transposed* and divided by the determinant to give

$$A^{-1} = \frac{1}{\det(A)} A_{\text{cof}}.$$

If $\det(A) = 0$, then A is not invertible.

Applications: Solution of $Ax = b$

Solving $Ax = b$, we have

$$x = A^{-1}b = \frac{1}{\det(A)} A_{\text{cof}} b.$$

It turns out (show this by expanding $\det(B_j)$ in cofactors of the jth column) that the matrix-vector product $A_{\text{cof}} b$ can be written in a convenient way, and we obtain

Cramer's Rule

$$x_j = \frac{\det(B_j)}{\det(A)},$$

where in B_j the vector b replaces the jth column of A.

Example 1 — Calculation of Determinant

$$
\begin{aligned}
|A| &= \begin{vmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix} \\
&= 3\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \\
&= 3(4+3) + 2(2+12) + 2(1-8) = 35
\end{aligned}
$$

Example 2 — Calculation of Inverse

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = (3)(12-1) - (2)(8-1) + (1)(2-3) = 18$$

$$A_{11} = (-1)^{1+1} \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} = 11 \qquad A_{12} = (-1)^{1+2} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = -7$$

$$A_{13} = (-1)^{1+3} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = -1$$

$$\mathbf{A}^{-1} = \frac{1}{18} \begin{bmatrix} 11 & -7 & -1 \\ -7 & 11 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

Example 3 — Cramer's rule

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 11 \\ 13 \\ 12 \end{bmatrix}$$

Let $D \equiv \det A, D_j \equiv \det B_j$.

$$D = (3)(12-1) - (2)(8-1) + (1)(2-3) = 18$$

D$\neq$ 0, **b**$\neq$ 0, $\Rightarrow$ unique non-trivial solution

$$D_1 = (11)(12-1) - (2)(52-12) + (1)(13-36) = 18$$
$$D_2 = (3)(52-12) - (11)(8-1) + (1)(24-13) = 54$$
$$D_3 = (3)(36-13) - (2)(24-13) + (11)(2-3) = 36$$

$$\Rightarrow x_1 = 1 \qquad x_2 = 3 \qquad x_3 = 2$$