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Algebraic Equations: no independent variable; involve only relations between dependent variables

Population Growth Model — Logistic Equation

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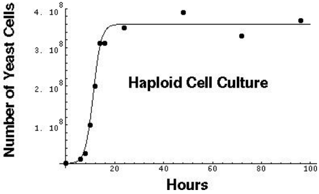

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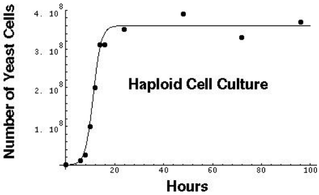

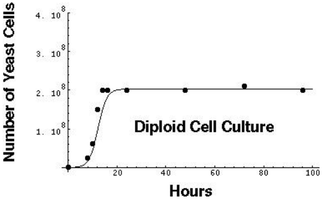

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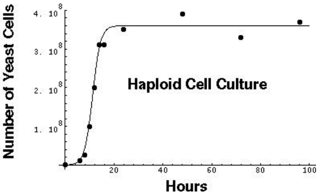

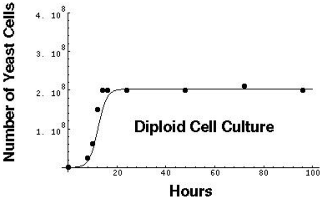

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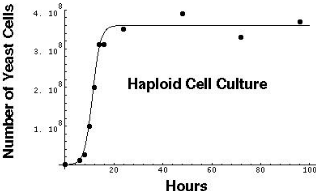

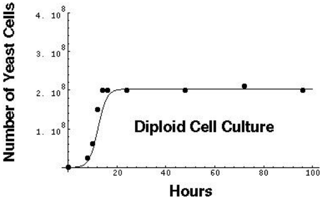

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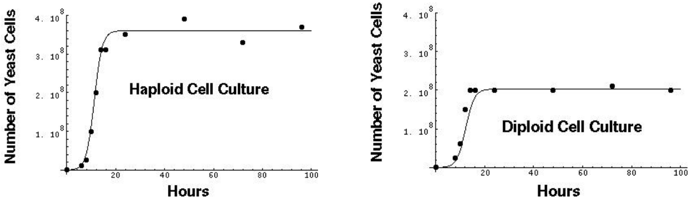

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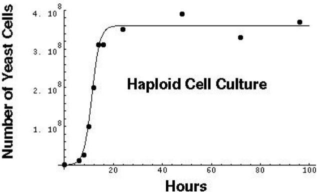

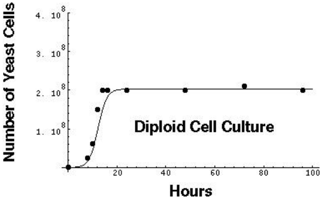

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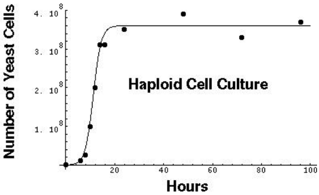

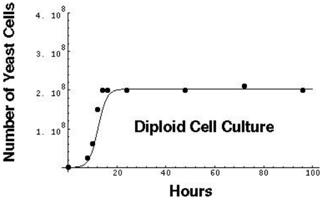

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Age of fossil $t = -(1.213 \times 10^{-4})^{-1} \cdot \ln 0.1 = 19000$ years

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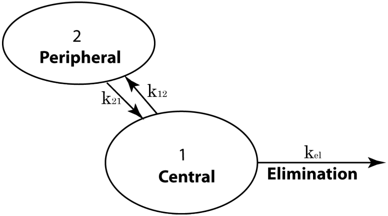
We will discuss later more sophisticated models :

e.g. the **S**(susceptible)-**I**(infected)-**R**(recovered) models, that include the rate of recovery from the infection.

Two-compartment Pharmacokinetic models



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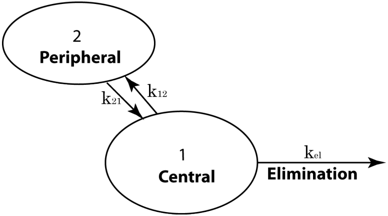

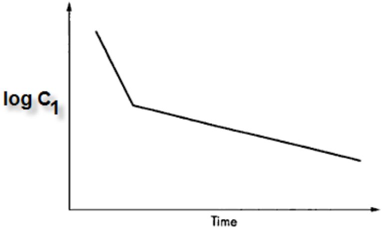

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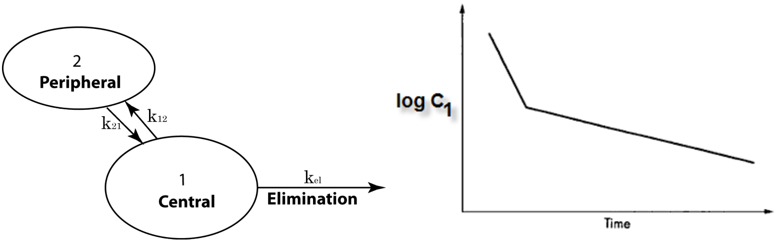

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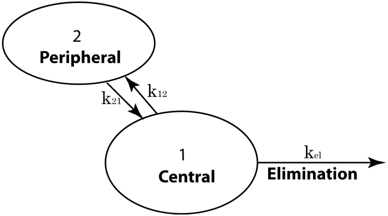

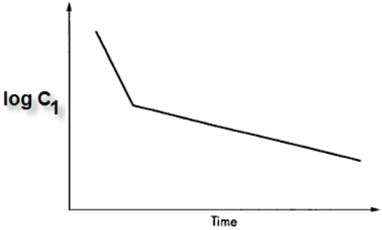

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[1]The energy is taken to consist of the sum of the kinetic, potential, and internal energy.

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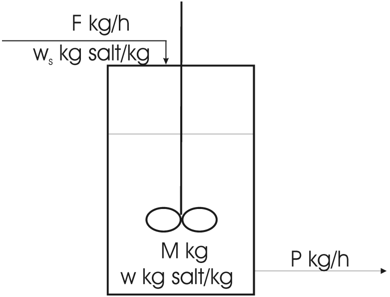

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We have two differential equations in seven variables: M, F, P, w, w_s, w_0, M_0. This can be solved if five of the variables are specified,e.g. F=10 kg/h, P=5 kg/h, w_s=0.1 kg salt/kg, as well as the initial conditions for the differential equations: M_0=100 kg, w(t=0)=w_0=0 (pure water).

Boundary-Value ODE Problem

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Linear System of Algebraic Equations

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Required to solve this system of equations for x_1, x_2, and x_3.

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Note: In terms of the fundamental units of mass [M], length [L] and time [t], the dimensions of viscosity are [M]/[L][t] and pressure are $[M]/[L][t]^2$. You should similarly write down the dimensions of the other physical variables in the given equation and equate the dimensions of the fundamental units on both sides of the equation.

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A system of linear algebraic equations in the exponents α, β, γ, and δ.

In-class exercise (1/20/10)

A lake has a volume of 500×10^9 m^3 with a pollutant mass fraction of x=0.0005. There is an annual inflow of 50×10^9 m^3 due to runoff ($x_{in} = 0.0002$), annual rainfall of 100×10^9 m^3, an annual evaporation loss of 100×10^9 m^3, and an annual drawoff for irrigation of 50×10^9 m^3.
What is the mass fraction of pollutant in the lake after two years?
You may assume that the lake contents are uniform.
Liquid density is 10^3 kg/m^3.