Lecture 2, Outline

- **Gaussian elimination**
- **Matrix multiplication**
- **LU factorization**
- **Inverse matrix and Gauss-Jordan elimination**
- **Ill-conditioned matrices and round-off errors**

SUGGESTED READING:

G. Strang, Linear algebra and its applications, Chapter 1

Gaussian Elimination, Geometry, Mechanics & Cost

- Reduce $N \times N$ system by repeatedly subtracting multiples of one equation from another equation
- Geometry: intersection of n subspaces ("hyper-planes"), singular cases
- Mechanics: forward elimination (clear out columns below pivots, $A \to U$) and back-substitution. Stop and think when zero pivot encountered (Row exchange? Look *below the zero pivot* for non-zero entry. Failure signifies singularity, i.e. no solution of ∞ of solutions.)
- Cost: FE $\approx \frac{1}{3}n^3$, BS $\approx \frac{1}{2}n^2$.

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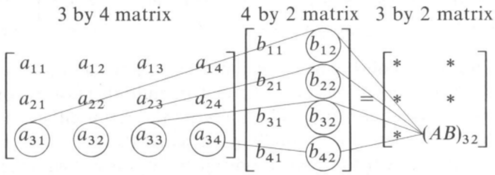
Matrix-Vector and Matrix-Matrix multiplication

- Inner product of two vectors
- Ax: by rows, by columns
- AB: (i,j) entry is the inner product of ith row of A and jth column of B. Why do it this way? Composition of linear functions, $h = f \circ g$.
- Running time $O(n^2)$, Strassen $O(n^{\log_2 7})$.



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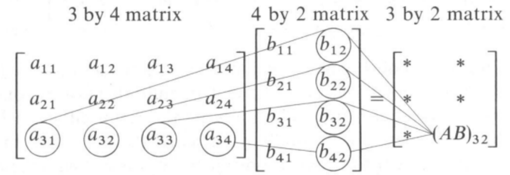

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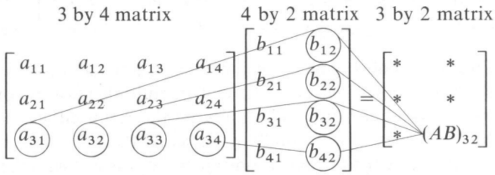

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$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix} \textit{(permutation)}$$

LU Factorization

- **Elementary matrix** subtracts a multiple l of row j from row i: 1's on the diagonal, (i,j) entry is $-l$.

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Gaussian elimination as GFE, applied to A,

$$GFE = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & 1 & 1 \end{bmatrix}$$

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"Reverse" Gaussian elimination as $E^{-1}F^{-1}G^{-1}$, applied to U,

$$E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix},$$

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$$L \equiv E^{-1}F^{-1}G^{-1} \Rightarrow \boxed{LU = A},$$

where U contains <u>pivots</u> and L contains <u>multipliers from GE</u>.

LU Factorization, comments

- A record of elimination steps, gives complete information.
- For multiple right hand sides, solve: $Lc = b$, then $Ux = c$, with $\frac{n^2}{2}$ operations each!
- How to write LU as LDU, where L and U have 1's on the diagonal and D is the diagonal matrix of pivots?

$$U = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{12}/d_1 & \dots \\ & 1 & u_{23}/d_2 & \dots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

- For a positive definite matrix (pivots positive), LDU reduces to RR^T (Cholesky factorization), where R is upper triangular.
- Not all non-singular matrices possess LU factors (leading principal sub-matrices must non-singular).

LU Factorization, examples

- With the rows reordered in advance, PA can be factored into LU, where P is a permutation matrix (how to find it?), e.g.

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad P_{23} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

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Find LU factors of $A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{bmatrix}$. *To save computer memory, A can be successively overwritten with information in L and U, as GE evolves.*

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<u>EXAMPLE</u>

Use partial pivoting on $A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 4 & 8 & 12 & -8 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{bmatrix}$ *and determine the LU decomposition* $PA = LU$*, where P is the associated permutation matrix. Hint: Adjoin a "permutation counter column" to keep track of row interchanges and remember to permute b as well, if available.*

Inverses and Transposes

- A invertible, if $A^{-1}A = AA^{-1} = I$
- Simple for 2 by 2 matrix, or diagonal matrix
- $(AB)^{-1} = B^{-1}A^{-1}$, similarly for three or more matrices
- Why bother? $Ax = b \Rightarrow x = A^{-1}b$
- Calculation of A^{-1}: Gauss-Jordan elimination

  Instead of stopping at U, we continue by subtracting multiples of a row from the rows above, until we reach I:

  $$\boxed{[A|I] \rightarrow [U|L^{-1}] \rightarrow [I|A^{-1}]}$$

  Not practical though (n^3 operations, sensitive to round-off errors).
- Multiple tests for invertibility: independent rows/columns, nonzero pivots/determinant/eigenenvalues, A^TA positive definite, full rank
- Transpose, $(AB)^T$, symmetric matrix

Ill-conditioned matrices and round-off errors

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- Ill-conditioned matrix is sensitive to small perturbations. Small pivots VS zero pivots. Condition number $\kappa = \| A \| \| A^{-1} \|$.

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$A = \begin{bmatrix} 1. & 1. \\ 1. & 1.0001 \end{bmatrix}$. *(ill-conditioned)*

For $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, *while for perturbed* $b = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *!*

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$$A' = \begin{bmatrix} 0.0001 & 1. \\ 1. & 1. \end{bmatrix}.$$ *(well-conditioned)*

For $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$*, solve with roundoff to 3 places. Then try <u>partial pivoting</u>, i.e. exchange rows so as to maximize the magnitudes of pivots.*