

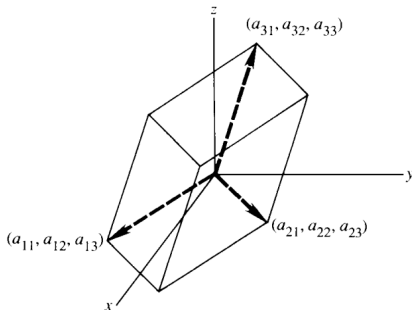
Lecture 5 (Determinants), Outline

- ▶ **Usefulness**
- ▶ **Properties and Definition**
- ▶ **Applications: Inverse Matrix and Cramer's Rule**
- ▶ **Examples**

SUGGESTED READING:

G. Strang, Linear algebra and its applications, Chapter 4

- ▶ Test for invertibility: **If determinant of A is zero, then A is singular.**
- ▶ **Volume of a parallelepiped** (edges come from rows/columns of A)



- ▶ **Formula for pivots:** determinant = \pm (product of pivots)

Properties

- 1 Determinant changes sign when two rows are exchanged.
- 2 $\det(I) = 1$.
- 3 If two rows of A are equal, $\det(A) = 0$.
- 4 If A contains a zero row, $\det(A) = 0$.
- 5 Subtracting a multiple of one row from another row leaves the determinant unchanged.
- 6 If A is triangular, $\det(A) = a_{11} a_{22} \cdots a_{nn}$.
- 7 $\det(AB) = \det(A)\det(B)$.
- 8 $\det(A^T) = \det(A)$.

To illustrate...

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned}
 &= \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} \\
 &+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix} \\
 &= \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}
 \end{aligned}$$

$n!$ permutations of rows/columns

- ▶ Two common definitions: expansion in cofactors (Laplace) or permutations of elements (Leibniz)

Expansion in Cofactors

The **determinant** of A is a combination of row i and the cofactors of row i (or, equivalently, column j and the cofactors of column j):

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

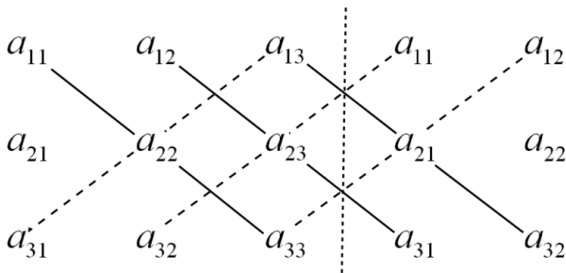
The **cofactor** A_{ij} is the determinant of M_{ij} with the correct sign:

$$A_{ij} = (-1)^{i+j} \det(M_{ij}).$$

The **minor** M_{ij} is formed by deleting row i and column j of A .

Sarrus' Rule

- ▶ Based on Leibniz definition of determinant
- ▶ A useful memorization scheme to calculate determinants of 3×3 matrices



Applications: Inverse matrix

Writing the cofactor expansion in matrix form, we obtain

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_A \underbrace{\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}}_{A_{\text{cof}}} = \underbrace{\begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix}}_{\det(A)I}$$

Multiplying the above by $\frac{A^{-1}}{\det(A)}$, we obtain a formula for A^{-1} :

Inverse matrix

The entries of A^{-1} are the cofactors of A , *transposed* and divided by the determinant to give

$$A^{-1} = \frac{1}{\det(A)} A_{\text{cof}}^T.$$

If $\det(A) = 0$, then A is not invertible.

Applications: Solution of $Ax = b$

Solving $Ax = b$, we have

$$x = A^{-1}b = \frac{1}{\det(A)} A_{\text{cof}} b.$$

It turns out (show this by expanding $\det(B_j)$ in cofactors of the j th column) that the matrix-vector product $A_{\text{cof}} b$ can be written in a convenient way, and we obtain

Cramer's Rule

$$x_j = \frac{\det(B_j)}{\det(A)},$$

where in B_j the vector b replaces the j th column of A .

Example 1 — Calculation of Determinant

$$|A| = \begin{vmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix}$$

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$$\mathbf{A}^{-1} = \frac{1}{18} \begin{bmatrix} 11 & -7 & -1 \\ -7 & 11 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

Example 3 — Cramer's rule

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Let $D \equiv \det A$, $D_j \equiv \det B_j$.

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$$D_3 = (3)(36 - 13) - (2)(24 - 13) + (11)(2 - 3) = 36$$

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$$\Rightarrow x_1 = 1 \quad x_2 = 3 \quad x_3 = 2$$