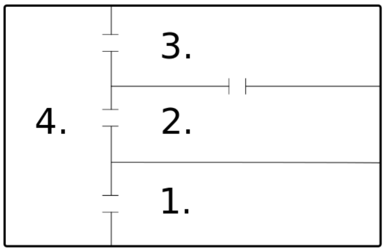
ECH3301 Process Analysis and Design Fall 2010

HW 2 N×N, M×N systems Due September 8

1. **(Matrix multiplication)** For $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix}$, determine $\lim_{n\to\infty} \mathbf{A}^n$.

   [Hint: Compute a few powers of $\mathbf{A}$ and try to deduce the general form of $\mathbf{A}^n$.]

2. **(N × N system)** Suppose that 100 insects are distributed in an enclosure consisting of four chambers with passageways between them as shown below.

   At the end of one minute, the insects have redistributed themselves. Assume that a minute is not enough time for an insect to visit more than one chamber and that at the end of a minute 40% of the insects in each chamber have not left the chamber they occupied at the beginning of the minute. The insects that leave a chamber disperse uniformly among the chambers that are directly accessible from the one they initially occupied. If at the end of one minute there are 12, 25, 26 and 37 insects in chambers 1., 2., 3. and 4., respectively, determine what the initial distribution had to be.

3. **(Hilbert matrix)** Let $f(x) = \sin \pi x$ on $[0, 1]$. The objective is to determine the coefficients α_i of the cubic polynomial $p(x) = \sum_{i=0}^{3} \alpha_i x^i$ that is as close to $f(x)$ as possible in the sense that

$$r \equiv \int_0^1 [f(x) - p(x)]^2 dx = \int_0^1 [f(x)]^2 dx - 2\sum_{i=0}^{3} \alpha_i \int_0^1 x^i f(x) dx + \int_0^1 \left(\sum_{i=0}^{3} \alpha_i x^i \right)^2 dx$$

   is as small as possible.

   a) In order to minimize r, impose the condition that $\frac{\partial r}{\partial \alpha_i} = 0$ for each $i = 0, 1, 2, 3$ and show this results in a system of linear equations whose augmented matrix is $[H_4|b]$, where

$$H_4 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \frac{2}{\pi} \\ \frac{1}{\pi} \\ \frac{1}{\pi} - \frac{4}{\pi^3} \\ \frac{1}{\pi} - \frac{6}{\pi^3} \end{bmatrix}.$$

   H_4 is called Hilbert matrix of order 4.

   b) Systems involving Hilbert matrices are badly ill-conditioned. Use exact arithmetic with Gaussian elimination to reduce H_4 to triangular form.

4. **(M × N system)** Determine the general solution of the following homogeneous system:

$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &= 0, \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0, \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0. \end{aligned}$$

5. **(Inverse matrix)** Find the matrix X such that $X = AX + B$, where

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}.$$

6. **(Linear independence)** Is the following set of vectors linearly independent? If not, write one of the vectors as a linear combinations of others.

$$\{(1 \quad 2 \quad 3), (0 \quad 4 \quad 5), (0 \quad 0 \quad 6), (1 \quad 1 \quad 1)\}.$$

[Hint: To write one vector as a combination of others in a dependent set, place the vectors as columns in A and find the echelon form. This reveals the dependence relationships among columns of A.]