Green's function for Stokes flow in a rectangular channel

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Abstract

In this work we develop a semi-analytical form of the Stokes flow Green's function in a rectangular channel using a combination of Fourier transform, eigenfunction expansions and a rapidly convergent representation of the Green's function for a potential flow. The solution may serve as a basis for further analytical study of flow disturbances and their hydrodynamic interaction with the confining walls, with potential use e.g. in the hydrodynamics of swimming microorganisms.

Keywords: *analytical Stokeslet, rectangular channel, Neuber-Papkovich formalism, finite Fourier transform, eigenfunction expansions*

1 Introductory remarks

In this work we derive an analytical form of the Stokeslet in a rectangular channel using a combination of Fourier transform, eigenfunction expansions and a rapidly convergent representation of the Green's function for a potential flow. To the best of our knowledge, we are not aware of any previous analytical solution to this problem. The solution may provide a basis for further analytical study of flow singularities and their interaction with the walls in strong confinement, with potential use e.g. in the hydrodynamics of swimming microorganisms.

The point force that represents the disturbing action of a bead on the solvent can be modeled by the Dirac delta function, giving rise to fundamental

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solutions of Stokes equation (SE), combined with the incompressibility constraint. While a general domain necessitates numerical solution, analytical Green's function (GF) can be readily obtained in free space [7] (Stokeslet) and with some difficulty in simple domains, such as in the vicinity of a plane wall [1], two parallel plane walls [4] and a corner [8, 9]. The analytical approach is typically [1, 4] based on the decomposition of the GF into a free space part, a collection of judiciously placed images of the original source points, and a complementary solution. As the GF is required to vanish at the boundary of the domain, the image system is constructed so as to cancel the majority of the components of the dyadic GF at the boundary, thereby making the subsequent analysis tractable. To cancel the remaining components, the standard SE (i.e. without Dirac delta forcing) is solved subject to the no-slip boundary condition simplified by the image system, yielding the complementary solution. For a single wall located at $x_2 = 0$, the image system consists of a set of three singularities: a Stokeslet S and two of its derivatives, namely Stokeslet doublet S^D and potential dipole P^D, defined as [6]

$$S_{ij}(\mathbf{r}) = \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}, \tag{1}$$

$$S^D_{ij}(\mathbf{r}) = \pm\frac{\partial S_{i2}}{\partial x_j} = x_2 P^D_{ij}(\mathbf{r}) \pm \frac{\delta_{j2}x_i - \delta_{i2}x_j}{r^3}, \tag{2}$$

$$P^D_{ij}(\mathbf{r}) = \pm\frac{\partial}{\partial x_j}\frac{x_i}{r^3} = \pm\left(\frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5}\right), \tag{3}$$

where $r = |\mathbf{r}|$ and the minus sign used for the x_2 direction, $j = 2$. In a slit, a superposition of the two single-wall solutions often gives sufficient approximation. A complete, yet slowly covering solution based on an infinite array of images is also available [4].

Rather than solving the complementary SE directly, an alternative approach [8, 9] is based on the Papkovich-Neuber formalism adopted from the linear elasticity theory, which allows the solution to the SE to be expressed in terms of harmonic functions, thereby transforming the problem to that of potential theory. Adopting the latter approach, we develop an analytical form of the Stokeslet in a rectangular channel using Fourier transform and eigenfunction expansions while building upon a rapidly convergent representation of the potential flow GF. The results are relevant not only in the context of Brownian dynamics studies of hydrodynamic interactions in microfluidic channels, but also in the biological study of fluid disturbances due to slender body motion, such as beating of cilia or flagella [1, 5] represented by lines of force singularities.

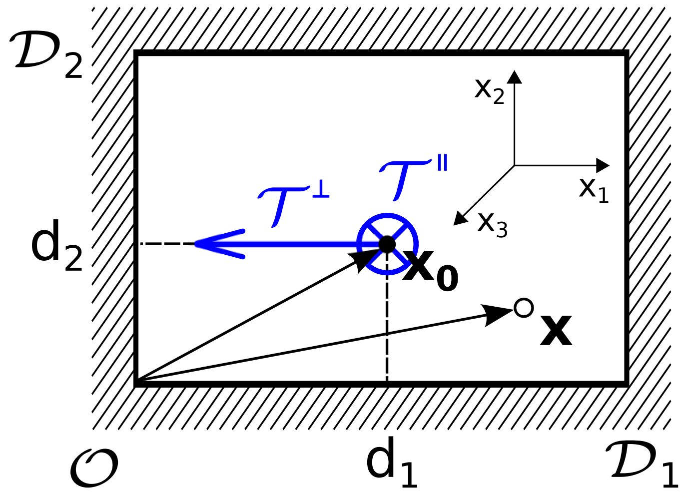

Fig. 1: Cross-section of an infinite rectangular channel of dimensions D_1 and D_2. The source point and field point are located at $\boldsymbol{x}_0 = [d_1, d_2, 0]$ and $\boldsymbol{x} = [x_1, x_2, x_3]$, respectively. General motion of the source point (T) is decomposed into the translation parallel ($T^{\parallel}$) and perpendicular ($T^{\perp}$) to the walls, $T = T^{\parallel} + T^{\perp}$.

2 Formulation and setup

Using Cartesian coordinates, our domain, depicted in Fig. 1, is a rectangular channel with walls at $x_1 = \{0, D_1\}$ and $x_2 = \{0, D_2\}$ and infinite axial dimension x_3. The source point is located at $\boldsymbol{x}_0 = [d_1, d_2, 0]$ and we observe the disturbance at $\boldsymbol{x} = [x_1, x_2, x_3]$. In the following we consider the translation of a particle parallel and perpendicular to the walls separately; the general motion in any direction can then be obtained by superposition.

3 Motion parallel and perpendicular to the walls

We first consider particle motion parallel to the walls of the channel. The governing Stokes equation for fluid with viscosity η and a point force acting in the x_3-direction and located at $\boldsymbol{x}_0$ is

$$\eta \nabla^2 \boldsymbol{v} + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\boldsymbol{e}_3 = \nabla p, \tag{4}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{5}$$

where $\delta(\boldsymbol{x})$ is the three-dimensional Dirac delta function, $\boldsymbol{e}_3$ is the unit vector in x_3-direction and the divergenceless velocity follows from the continuity equation. By virtue of the no-slip boundary condition, the velocity is required to vanish on the walls,

$$\boldsymbol{v} = 0 \quad \text{on the walls.} \tag{6}$$

The Papkovich-Neuber solution [10], based on the analogy between the Stokes flow and Cauchy equation governing small deformations of incompressible elastic materials, can be written as

$$\boldsymbol{v} = \nabla(\boldsymbol{x} \cdot \boldsymbol{\phi} + \omega) - 2\boldsymbol{\phi}, \qquad p = 2\eta\nabla \cdot \boldsymbol{\phi}, \tag{7}$$

with harmonic functions ω and $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ satisfying Laplace equations,

$$\nabla^2\omega = \nabla^2\phi_n = 0, \qquad n = 1, 2, 3. \tag{8}$$

In a star-shaped domain D (i.e. a domain for which $\exists x_0 \in D \subset \mathbf{R}^3 : \forall x \in D$, the line segment $\overline{x_0 x} \in D$), the scalar harmonic ω is optional [10] and is discarded for the motion parallel to the walls.

First we determine ϕ_3, defining $\chi \equiv 2\eta\phi_3$, where the harmonic χ must represent a point source located at $\boldsymbol{x}_0$ and acting in x_3-direction. We thus require that χ be governed by

$$\nabla^2\chi = \delta(x_1 - d_1)\delta(x_2 - d_2)\delta(x_3), \tag{9}$$

$$\chi = 0 \text{ on the walls,} \tag{10}$$

yielding a Green's function associated with potential flow. In free space, we would simply have $\chi_{FS} = 1/(4\pi r)$, where $r^2 = (x_1 - d_1)^2 + (x_2 - d_2)^2 + x_3^2$, and $\phi_1 = \phi_2 = 0$. Near a corner (at the intersection of two perpendicular plane walls), χ can be written [8] as a sum of four terms: χ_{FS} due to the original source in free space and three corrections due to images with respect to both walls and the origin. In a slit (two parallel planes) [4], the sum will now contain an infinite array of periodic images in the direction normal to the walls. In a rectangular channel, a double sum appears in the image system due to two periodic arrays of images, placed in directions normal to the walls [4]. We can then write χ in terms of the periodic Green's function G,

$$\chi(\boldsymbol{x}, \boldsymbol{x}_0) = G(x_1 - d_1, x_2 - d_2, x_3) + G(x_1 + d_1, x_2 + d_2, x_3) \tag{11}$$

$$- G(x_1 - d_1, x_2 + d_2, x_3) - G(x_1 + d_1, x_2 - d_2, x_3), \tag{12}$$

where $G(x_1, x_2, x_3)$ represents a source periodic in x_1 and x_2 directions with the period $2D_1$ and $2D_2$, respectively. The periodic Green's function is thus governed by the corresponding Laplace equation,

$$\nabla^2 G = \sum_m \delta(x_1 - 2D_1 m) \sum_n \delta(x_2 - 2D_2 n)\delta(x_3), \tag{13}$$

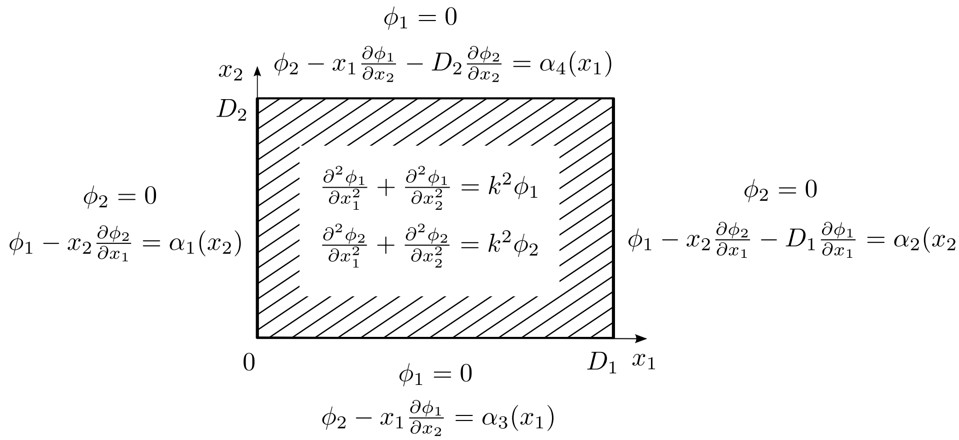

Fig. 2: The system of Helmholtz equations coupled through the boundary conditions on a rectangle $D_1 \times D_2$.

where $\sum_m$ denotes the sum over all integers m. The solution can be given in several forms, equivalent mathematically but not computationally due to different convergence properties. While the sum of images exhibits extremely poor convergence, a rapidly convergent form is available in terms of Fourier series [3],

$$G = \frac{|x_3|}{8D_1D_2} - \frac{1}{4\pi} \sum_{m,n}{}' \frac{\mathrm{e}^{-(\alpha_n^2+\beta_m^2)|x_3|}}{d_{mn}} \mathrm{e}^{i(\beta_m x_1 + \alpha_n x_2)}, \tag{14}$$

where $\sum'_{m,n}$ denotes the sum over all pairs of integers (m, n) except $(0, 0)$, $\alpha_n = \frac{n\pi}{D_2}$, $\beta_m = \frac{m\pi}{D_1}$ and $d_{mn} = \frac{2D_1D_2}{\pi}\left(\alpha_n^2 + \beta_m^2\right)^{1/2}$. This form is convenient for later differentiation with respect to x_1 and x_2 and provides fast convergence given sufficient distance between the field point and the source point along the channel direction, which can be assumed in pressure-driven or electrically-driven flows for moderate Wi or Pe.

To complete the solution, we are left to compute a mixed type boundary value problem on a rectangular domain for harmonic functions ϕ_1 and ϕ_2, with the no-slip boundary conditions on the walls, Fig. 2. Taking the Fourier transform of Eq. 8 with respect to x_3, i.e.

$$\mathcal{F}[f] = \hat{f}(x_1, x_2, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx_3 \exp(ikx_3) f(x_1, x_2, x_3), \tag{15}$$

we obtain an eigenvalue system

$$\frac{\partial^2 \hat{\phi}_1}{\partial x_1^2} + \frac{\partial^2 \hat{\phi}_1}{\partial x_2^2} = k^2 \hat{\phi}_1$$
$$\frac{\partial^2 \hat{\phi}_2}{\partial x_1^2} + \frac{\partial^2 \hat{\phi}_2}{\partial x_2^2} = k^2 \hat{\phi}_2, \quad (16)$$

with the transformed boundary conditions (Eq. 6) given as

$$\hat{\phi}_1 = 0, \qquad \hat{\phi}_2 - x_1 \frac{\partial \hat{\phi}_1}{\partial x_2} = \hat{\alpha}_3(x_1) \qquad \text{at } x_1 = 0,$$
$$\hat{\phi}_1 = 0, \qquad \hat{\phi}_2 - x_1 \frac{\partial \hat{\phi}_1}{\partial x_2} - D_2 \frac{\partial \hat{\phi}_2}{\partial x_2} = \hat{\alpha}_4(x_1) \qquad \text{at } x_1 = D_2,$$
$$\hat{\phi}_2 = 0, \qquad \hat{\phi}_1 - x_2 \frac{\partial \hat{\phi}_2}{\partial x_1} = \hat{\alpha}_1(x_2) \qquad \text{at } x_2 = 0,$$
$$\hat{\phi}_2 = 0, \qquad \hat{\phi}_1 - x_2 \frac{\partial \hat{\phi}_2}{\partial x_1} - D_1 \frac{\partial \hat{\phi}_1}{\partial x_1} = \hat{\alpha}_2(x_2) \qquad \text{at } x_2 = D_1, \quad (17)$$

where $\hat{\alpha}_i = \mathcal{F}[\alpha_i]$, with $\alpha_{1,2} = x_3 \frac{\partial \chi}{\partial x_1}\Big|_{x_1=0,D_1}$ and $\alpha_{3,4} = x_3 \frac{\partial \chi}{\partial x_2}\Big|_{x_2=0,D_2}$. The simplified boundary conditions, Eq. 17, were derived based on the assumption that at the walls $x_i = 0$ $(i = 1, 2)$, ϕ_n for $n = 1, 2, 3$ are functions of x_i only, which can be assumed if we interpret a solid wall as a limiting streamline and note the orthogonality between streamlines and equipotential lines of the velocity.

Having now homogeneous boundary conditions for ϕ_1 and ϕ_2 in x_2 and x_1 directions, respectively, we use finite Fourier transforms (eigenfunction expansions) [2] and expand in the mentioned directions, with the basis functions θ_n^i provided by the corresponding Sturm-Liouville problems. The expansions are also substituted in Eq. 17, arriving at the following expression for ϕ_1:

$$\hat{\phi}_1 = \sum_n c_n(x_1) \theta_n^2(x_2), \quad (18)$$

$$c_n(x_1) = A_n f_n(1 - x_1) + B_n f_n(x_1), \quad (19)$$

$$A_n = \int_0^{D_2} dx_2\, \theta_n^2(x_2) \left[\hat{\alpha}_1(x_2) + x_2 \sum_m d_m(x_2) Q_m^1 \right] \quad (20)$$

$$B_n = \int_0^{D_2} dx_2\, \theta_n^2(x_2) \left[\hat{\alpha}_2(x_2) + x_2 \sum_m d_m(x_2)(-1)^m Q_m^1 + D_1 \sum_m \frac{dc_m}{dx_1}\Big|_{x_1=D_1} \theta_m^2(x_2) \right], \quad (21)$$

where $n = 1, 2, \ldots$. Similarly for ϕ_2, we have:

$$\hat{\phi}_2 = \sum_n d_n(x_2)\theta_n^1(x_1), \tag{22}$$

$$d_n(x_2) = M_n f_n(1 - x_2) + N_n f_n(x_2), \tag{23}$$

$$M_n = \int_0^{D_1} dx_1\, \theta_n^1(x_1) \left[\hat{\alpha}_3(x_1) + x_1 \sum_m c_m(x_1) Q_m^2\right] \tag{24}$$

$$N_n = \int_0^{D_1} dx_1\, \theta_n^1(x_1) \left[\hat{\alpha}_4(x_1) + x_1 \sum_m c_m(x_1)(-1)^m Q_m^2 + D_2 \sum_m \left.\frac{dd_m}{dx_2}\right|_{x_2=D_2} \theta_m^1(x_1)\right], \tag{25}$$

where the superscripts $i = 1, 2$ distinguish between equivalent formulas for D_1 and D_2, respectively. We thus have $Q_n^i = \sqrt{\frac{2}{D_i}}\frac{n\pi}{D_i}$ and the basis functions $\theta_n^i(x) = \sqrt{\frac{2}{D_i}} \sin \frac{n\pi x}{D_i}$. The function f_n has the form

$$f_n(x) = \frac{\sinh s_n x}{\sinh s_n}, \tag{26}$$

where $s_n = \sqrt{k^2 + (n\pi)^2}$.

Eqs. 20-21 and 24-25 represent an infinite system of coupled summation equations for A_n, B_n, M_n and N_n, effectively coupling the Fourier coefficients c_n and d_n defined in Eqs. 19 and 23, respectively. To proceed further, the system needs to be decoupled by taking partial sums, which can be justified by fast convergence of the eigenfunction expansions for $\hat{\phi}_1, \hat{\phi}_2$ and ϕ_3. Upon truncating the sums at N terms (where N is the number of Fourier coefficients we wish to retain) and further rearrangement, we obtain a linear system

$$\boldsymbol{z} = \boldsymbol{\Gamma} \cdot \boldsymbol{z} + \boldsymbol{p}, \tag{27}$$

where vector $\boldsymbol{z}$ contains $4N$ elements with $\boldsymbol{z_n}$ denoting the 4 unknowns A_n, B_n, M_n and N_n, $n = 1 \ldots N$. Thus for the $\boldsymbol{z_n}$ component we can write $\boldsymbol{z}_n = \sum_{m=1}^N \boldsymbol{\Gamma}_{nm}\boldsymbol{z}_m + \boldsymbol{p}_n$. The $\boldsymbol{p}_n$-component of the $(4N \times 1)$ vector $\boldsymbol{p}$ and the $\boldsymbol{\Gamma}_{nm}$ submatrix of the $(4N \times 4N)$ block matrix $\boldsymbol{\Gamma}$ are given by, respectively,

$$\boldsymbol{p}_n = \begin{bmatrix} p_n^{12} \\ p_n^{22} \\ p_n^{31} \\ p_n^{41} \end{bmatrix}, \tag{28}$$

$$\mathbf{\Gamma}_{nm} = \begin{bmatrix} 0 & 0 & h^{12}_{nm}(1-x_2) & h^{12}_{nm}(x_2) \\ -k^1_{nm}(1-D_1) & k^1_{nm}(D_1) & (-1)^m(\Gamma_{nm})_{13} & (-1)^m(\Gamma_{nm})_{14} \\ h^{21}_{nm}(1-x_1) & h^{21}_{nm}(x_1) & 0 & 0 \\ (-1)^m(\Gamma_{nm})_{31} & (-1)^m(\Gamma_{nm})_{32} & -k^2_{nm}(1-D_2) & k^2_{nm}(D_2) \end{bmatrix}, \tag{29}$$

where we define

$$\begin{aligned} p^{ij}_n &= \int_0^{D_j} \theta^j_n \alpha_i dx_j, \\ h^{ij}_{nm}(y) &= Q^i_m \int_0^{D_j} \theta^j_n x_j f_m(y) dx_j, \\ k^j_{nm}(y) &= \delta_{nm} D_j s_n \cosh(y s_n) \operatorname{csch} s_n, \end{aligned} \tag{30}$$

with $\operatorname{csch}(x)$ the hyperbolic cosecant. Upon solving Eq. 27 for $\boldsymbol{z}$ we can compute the Fourier coefficients from Eqs. 19 and 23. Substituting the result into the eigenfunction expansions (Eqs. 18 and 22) yields $\hat{\phi}_1$ and $\hat{\phi}_2$ and the inverse Fourier transforms,

$$f(x_1, x_2, x_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \exp(ikx_3) \hat{f}(x_1, x_2, k), \tag{31}$$

then give the desired solutions ϕ_1 and ϕ_2.

We have thus computed the velocity due to a point force in x_3 direction, effectively recovering the third column of the dyadic Green's function $\mathbf{\Omega}$ for the Stokes flow in a rectangular channel. To compute the remaining two columns Ω_{i1} and Ω_{i2}, with $i = 1 \ldots 3$, we need to consider the motion in the directions x_1 and x_2. Assuming for convenience $D_1 = D_2$ (this assumption is nonessential), the solutions in both directions will be identical save for the interchange of x_1 and x_2 variables and d_1 and d_2 constants below. Hence we discuss only the translation in the negative x_1 direction.

With the hope of deriving a system analogous to Eq. 16-17, we use the full Papkovich-Neuber solution, Eq. 7, including ω. Following [8], we set $\omega = -d_1\chi$. The harmonic function ϕ_1 can then be written in terms of the free-space contribution χ and a correction analogous to the complementary solution in Blake [1] and Liron [4], which we denote ϕ_{1c}. We then have

$$\phi_1 = \chi + \phi_{1c}. \tag{32}$$

Recognizing that a point force not acting in the only unbounded direction x_3 will yield ϕ_3=0, we are left to solve a system identical to Eqs. 16-17 for

ϕ_{1c} and ϕ_2, with the correction ϕ_{1c} replacing the original ϕ_1 and functions α_i, $i = 1 \ldots 4$, modified as follows:

$$\alpha_{1,2} = -d_1 \frac{\partial \chi}{\partial x_1}\bigg|_{x_1=0,D_1}, \quad \alpha_{3,4} = \frac{\partial}{\partial x_2}[(x_1 - d_1)\chi]\bigg|_{x_2=0,D_2}. \tag{33}$$

The solution method is then identical to the one used for the particle motion parallel to the bounding walls.

References

[1] JR Blake, *A note on the image system for a Stokeslet in a no-slip boundary*, Proc. Camb. Phil. Soc, vol. 70, Cambridge Univ Press, 1971, pp. 303–310.

[2] W.M. Deen, *Analysis of transport phenomena. Topics in Chemical Engineering*, vol. 3, Oxford University Press, New York, 1998.

[3] CM Linton, *Rapidly convergent representations for Green'functions for Laplace'equation*, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences **455** (1999), no. 1985, 1767–1797.

[4] N. Liron and S. Mochon, *Stokes flow for a stokeslet between two parallel flat plates*, Journal of Engineering Mathematics **10** (1976), no. 4, 287–303.

[5] N. Liron and R. Shahar, *Stokes flow due to a Stokeslet in a pipe*, Journal of Fluid Mechanics **86** (1978), no. 04, 727–744.

[6] H. Ma and M.D. Graham, *Theory of shear-induced migration in dilute polymer solutions near solid boundaries*, Physics of Fluids **17** (2005), 083103.

[7] C. Pozrikidis, *Boundary integral and singularity methods for linearized viscous flow*, no. 8, Cambridge University Press, 1992.

[8] O. Sano and H. Hasimoto, *Slow motion of a spherical particle in a viscous fluid bounded by two perpendicular walls*, Journal of the Physical Society of Japan **40** (1976), no. 3.

[9] ———, *Slow Motion of a Small Sphere in a Viscous Fluid in a Corner I. Motion on and across the Bisector of a Wedge*, Journal of the Physical Society of Japan **42** (1977), no. 1, 306–312.

[10] T. Tran-Cong and JR Blake, *General solutions of the Stokes' flow equations*, Journal of Mathematical Analysis and Applications **90** (1982), no. 1, 72–84.